Integration Review Solutions

Prof. Leyda Almodóvar

Name: _____

1. Find the general indefinite integral.

   (a) $\int \sqrt[4]{x^3}\,\mathrm{d}x = \int x^{\frac{3}{4}}\,\mathrm{d}x = \boxed{\frac{4}{7}x^{\frac{7}{4}} + C}$

   (b) $\int \sqrt{t}(t^2+5t+3)\,\mathrm{d}t = \int t^{\frac{5}{2}} + 5t^{\frac{3}{2}} + 3t^{\frac{1}{2}}\,\mathrm{d}t = \boxed{\frac{2}{7}t^{\frac{7}{2}} + 2t^{\frac{5}{2}} + 2t^{\frac{3}{2}} + C}$

   (c) $\int (x^2 + 1 + \frac{1}{x^2+1})\,\mathrm{d}x = \boxed{\frac{x^3}{3} + x + \tan^{-1} x + C}$

   (d) $\int \sec t(\sec t + \tan t)\,\mathrm{d}t = \int (\sec^2 t + \sec t + \tan t)\,\mathrm{d}t = \boxed{\tan t + \sec t + C}$

2. Evaluate the integral.

(a) $\int_1^2 (4x^3 - 3x^2 + 2x)\,dx = \Big[x^4 - x^3 + x^2\Big]_1^2 = 16 - 8 + 4 - (1 - 1 + 1) = \boxed{11}$

(b) $\int_{-1}^1 t(1-t)^2\,dt = \int_{-1}^1 t(1 - 2t + t^2)\,dt = \int_{-1}^1 (t - 2t^2 + t^3)\,dt = \Big[\frac{t^2}{2} - \frac{2t^3}{3} + \frac{t^4}{4}\Big]_{-1}^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \Big(\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\Big) = \boxed{-\frac{4}{3}}$

(c) $\int_1^2 (\frac{1}{x^2} - \frac{4}{x^3})\,dx = \int_1^2 (x^{-2} - 4x^{-3})\,dx = \Big[\frac{x^{-1}}{-1} - \frac{4x^{-2}}{-2}\Big]_1^2 = -\frac{1}{2} + \frac{2}{4} - (-1 + 2) = \boxed{-1}$

(d) $\int_0^1 \frac{4}{1+p^2}\,dp = 4\int_0^1 \frac{1}{1+p^2}\,dp = 4\Big[\tan^{-1} p\Big]_0^1 = 4\tan^{-1}(1) - 4\tan^{-1}(0) = 4\big(\frac{\pi}{4}\big) - 4(0) = \boxed{\pi}$

(e) $\int_1^4 \frac{\sqrt{y} - y}{y^2}\,dy = \int_1^4 \Big(y^{-\frac{3}{2}} - \frac{1}{y}\Big)\,dy = \Big[-2y^{-\frac{1}{2}} - \ln y\Big]_1^4 = -\frac{2}{\sqrt{4}} - \ln 4 - \Big(-\frac{2}{1} - \ln 1\Big) = -1 - \ln 4 + 2 = \boxed{1 - \ln 4}$

(f) $\int_0^{\frac{\pi}{4}} \frac{1+\cos^2\theta}{\cos^2\theta}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} (\sec^2\theta + 1)\,\mathrm{d}\theta = \Big[\tan\theta + \theta\Big]_0^{\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) + \frac{\pi}{4} - (\tan 0 + 0) = \boxed{1 + \frac{\pi}{4}}$

(g) $\int_0^1 (5x - 5^x)\,\mathrm{d}x = \left[\frac{5x^2}{2} - \frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{2} - \frac{5}{\ln 5} - \left(0 - \frac{1}{\ln 5}\right) = \boxed{\frac{5}{2} - \frac{4}{\ln 5}}$

3. Evaluate the indefinite integral (using substitution).

(a) $\int x^2 e^{x^3}\,\mathrm{d}x$

Let $u = x^3$. Then, $\mathrm{d}u = 3x^2\,\mathrm{d}x$, which implies $x^2\,\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$. Then,

$$\int x^2 e^{x^3}\,\mathrm{d}x = \int \tfrac{1}{3} e^u\,\mathrm{d}u = \tfrac{1}{3}e^u + C = \boxed{\frac{1}{3}e^{x^3} + C}$$

(b) $\int \sin t\sqrt{1+\cos t}\,\mathrm{d}t$

Let $u = 1 + \cos t$. Then, $\mathrm{d}u = -\sin t\,\mathrm{d}t$, which implies $\sin t\,\mathrm{d}t = -\,\mathrm{d}u$. Then,

$$\int \sin t\sqrt{1+\cos t}\,\mathrm{d}t = \int -\sqrt{u}\,\mathrm{d}u = -\tfrac{2}{3}u^{\frac{3}{2}} + C = \boxed{-\frac{2}{3}(1+\cos t)^{\frac{3}{2}} + C}$$

(c) $\int \sec^2 2\theta\,\mathrm{d}\theta$

Let $u = 2\theta$. Then, $\mathrm{d}u = 2\,\mathrm{d}\theta$, which implies $\mathrm{d}\theta = \frac{1}{2}\,\mathrm{d}u$. Then,

$$\int \sec^2 2\theta\,\mathrm{d}\theta = \int \tfrac{1}{2}\sec^2 u\,\mathrm{d}u = \tfrac{1}{2}\tan u + C = \boxed{\frac{1}{2}\tan 2\theta + C}$$

(d) $\int y^2(4-y^3)^{\frac{2}{3}}\,\mathrm{d}y$

Let $u = 4 - y^3$. Then, $\mathrm{d}u = -3y^2\,\mathrm{d}y$, which implies $y^2\,\mathrm{d}y = -\frac{1}{3}\,\mathrm{d}u$. Then,

$$\int y^2(4-y^3)^{\frac{2}{3}}\,\mathrm{d}y = \int -\tfrac{1}{3}u^{\frac{2}{3}}\,\mathrm{d}u = -\tfrac{1}{5}u^{\frac{5}{3}} + C = \boxed{-\frac{1}{5}(4-y^3)^{\frac{5}{3}} + C}$$

(e) $\int e^{-5r}\,\mathrm{d}r$

Let $u = -5r$. Then, $\mathrm{d}u = -5\,\mathrm{d}r$, which implies $\mathrm{d}r = -\frac{1}{5}\,\mathrm{d}u$. Then,

$$\int e^{-5r}\,\mathrm{d}r = \int -\tfrac{1}{5}e^u\,\mathrm{d}u = -\tfrac{1}{5}e^u + C = -\boxed{-\frac{1}{5}e^{-5r} + C}$$

(f) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$

Let $u = \sqrt{x}$. Then, $\mathrm{d}u = \frac{1}{2}x^{-\frac{1}{2}}\,\mathrm{d}x$, which implies $\frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\,\mathrm{d}u$. Then,

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x = 2\int \sin u\,\mathrm{d}u = -2\cos u + C = \boxed{-2\cos\sqrt{x} + C}$$

(g) $\int \sin x \sin(\cos x)\,\mathrm{d}x$

Let $u = \cos x$. Then, $\mathrm{d}u = -\sin x\,\mathrm{d}x$, which implies $\sin x\,\mathrm{d}x = -\,\mathrm{d}u$. Then,

$$\int \sin x \sin(\cos x)\,\mathrm{d}x = \int -\sin u\,\mathrm{d}u = \cos u + C = \boxed{\cos(\cos x) + C}$$

(h) $\int \sec^2\theta \tan^3\theta\,\mathrm{d}\theta$

Let $u = \tan\theta$. Then, $\mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta$. Then,

$$\int \sec^2\theta\tan^3\theta\,\mathrm{d}\theta = \int u^3\,\mathrm{d}u = \tfrac{u^4}{4} + C = \boxed{\frac{\tan^4\theta}{4} + C}$$

4. Evaluate the definite integral (using substitution).

(a) $\int_0^1 (3t-1)^{50}\,dt$

Let $u = 3t-1$. Then, $du = 3\,dt$, which implies $dt = \frac{1}{3}\,du$. If $t = 0$, then $u = 3(0)-1 = -1$ and if $t = 1$, then $u = 3(1)-1 = 2$. Then,

$$\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^2 \tfrac{1}{3}u^{50}\,du = \left[\tfrac{u^{51}}{51(3)}\right]_{-1}^2 = \tfrac{2^{51}}{153} - \tfrac{(-1)^{51}}{153} = \boxed{\frac{2^{51}}{153} + \frac{1}{153}}$$

(b) $\int_0^3 \frac{1}{5x+1}\,dx$

Let $u = 5x+1$. Then, $du = 5\,dx$, which implies $dx = \frac{1}{5}\,du$. If $x = 0$, then $u = 5(0)+1 = 1$ and if $x = 3$, then $u = 5(3)+1 = 16$. Then,

$$\int_0^3 \tfrac{1}{5x+1}\,dx = \int_1^{16} \tfrac{1}{5}\cdot\tfrac{1}{u}\,du = \tfrac{1}{5}\Big[\ln u\Big]_1^{16} = \tfrac{1}{5}\Big(\ln 16 - \ln 1\Big) = \boxed{\frac{\ln 16}{5}}$$

(c) $\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

Let $u = e^z + z$. Then, $du = (e^z+1)\,dz$. If $z = 0$, then $u = e^0 + 0 = 1$ and if $z = 1$, then $u = e^1 + 1$. Then,

$$\int_0^1 \tfrac{e^z+1}{e^z+z}\,dz = \int_1^{e+1} \tfrac{1}{u}\,du = \Big[\ln u\Big]_1^{e+1} = \ln(e+1) - \ln 1 = \boxed{\ln(e+1)}$$

(d) $\int_0^1 xe^{-x^2}\,dx$

Let $u = -x^2$. Then, $du = -2x\,dx$, which implies $x\,dx = -\frac{1}{2}\,du$. If $x = 0$, then $u = -0^2 = 0$ and if $x = 1$, then $u = -(1)^2 = -1$. Then,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} -\tfrac{1}{2}e^u\,du = \tfrac{1}{2}\int_{-1}^0 e^u\,du = \tfrac{1}{2}\Big[e^u\Big]_{-1}^0 = \tfrac{1}{2}\Big(e^0 - e^{-1}\Big) = \boxed{\frac{1}{2}\Big(1 - \frac{1}{e}\Big)}$$

Integration review

Let c, k, and C be constants. Then,

1. $\int cf(x) = c\int f(x)\,dx$
2. $\int k\,dx = kx + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
4. $\int e^x\,dx = e^x + C$
5. $\int \sin x\,dx = -\cos x + C$
6. $\int \cos x\,dx = \sin x + C$
7. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
9. $\int \sec x\tan x\,dx = \sec x + C$
10. $\int \csc x\cot x\,dx = -\csc x + C$
11. $\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$
12. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

$\int x^{-1}dx = \ln|x| + C$

$\int b^x dx = \frac{b^x}{\ln b} + C$

Example. Find $\int (10x^4 - 3\sec^2 x)\,dx$.

$\int 10x^4 - 3\sec^2 x\,dx = \frac{10x^5}{5} - 3\tan x + C$

$= \boxed{2x^5 - 3\tan x + C}$

Example. Find $\int \frac{\cos x}{\sin^2 x}\,dx$. $\neq \frac{\int \cos x\,dx}{\int \sin^2 x\,dx}$

Solution #1:

u-sub: $u = \sin^2 x$ $\quad u = \cos x$ $\quad u = \sin x$

$du = 2\sin x\cos x\,dx$ $\quad du = -\sin x\,dx$ $\quad du = \cos x\,dx$

$\int \frac{\cos x}{\sin^2 x}dx = \int \frac{1}{u^2}du = \int u^{-2}du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{\sin x} + C$

$= \boxed{-\csc x + C}$

Solution #2:

$\int \frac{\cos x}{\sin^2 x}dx = \int \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}dx = \int \cot x\csc x\,dx = \boxed{-\csc x + C}$

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Example. Find $\int \frac{7x^2 + 2x^2\sqrt{x} - 5}{x^2}\,dx$.

$$= \int \frac{7x^2}{x^2} + \frac{2x^2\sqrt{x}}{x^2} - \frac{5}{x^2}\,dx = \int 7 + 2\sqrt{x} - \frac{5}{x^2}\,dx$$

$$= \int 7 + 2x^{1/2} - 5x^{-2}\,dx = 7x + 2\frac{x^{3/2}}{3/2} - \frac{5x^{-1}}{-1} + C$$

$$= 7x + 2 \cdot \frac{2}{3} x^{3/2} + 5x^{-1} + C$$

$$= \boxed{7x + \frac{4}{3} x^{3/2} + 5x^{-1} + C}$$

Example. Find $\int \left(4x^3 - 7x + \frac{4}{x^2+1}\right) dx$.

$$\int 4x^3 - 7x + \frac{4}{x^2+1}\,dx = \boxed{x^4 - \frac{7x^2}{2} + 4\tan^{-1}x + C}$$

Example. Find $\int e^{5x}\,dx$.

$u = 5x$

$du = 5dx$

$\frac{1}{5}du = dx$

$$\int e^{5x}\,dx = \int e^u \cdot \frac{1}{5}\,du = \frac{1}{5}\int e^u\,du$$

$$= \frac{1}{5}e^u + C = \boxed{\frac{1}{5}e^{5x} + C}$$

$$\int \cos(7x)\,dx = \frac{\sin(7x)}{7} + C$$

Made with Goodnotes

Example. Find $\int_0^1 \sqrt{3x+1}\,dx$.

$u = 3x+1$

$du = 3dx$

$\frac{1}{3}du = dx$

$x=0,\ u = 3\cdot 0+1 = 1$

$x=1,\ u = 3\cdot 1+1 = 4$

$$\int_0^1 \sqrt{3x+1}\,dx = \int_1^4 \sqrt{u}\cdot\frac{1}{3}\,du$$

$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big]_1^4$$

$$= \frac{2}{9}\cdot 4^{3/2} - \frac{2}{9}\cdot 1^{3/2}$$

$$= \frac{2}{9}\cdot 8 - \frac{2}{9} = \boxed{\frac{14}{9}}$$

Example. Find $\int \frac{x}{\sqrt{1-9x^2}}\,dx$.

$u = 1-9x^2$

$du = -18x\,dx$

$-\frac{1}{18}du = x\,dx$

$$\int \frac{x}{\sqrt{1-9x^2}}\,dx = -\frac{1}{18}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{18}\int u^{-1/2}\,du = -\frac{1}{18}\,\frac{u^{1/2}}{1/2} + C$$

$$= -\frac{1}{18}\cdot 2\cdot\sqrt{u} + C = \boxed{\frac{-\sqrt{1-9x^2}}{9} + C}$$

Example. Evaluate $\int_1^e \frac{\ln x}{x}\,dx$.

$u = \ln x$

$du = \frac{1}{x}dx$

$x=1,\ u = \ln 1 = 0$

$x=e,\ u = \ln e = 1$

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1$$

$$= \frac{1}{2} - \frac{0}{2} = \boxed{\frac{1}{2}}$$

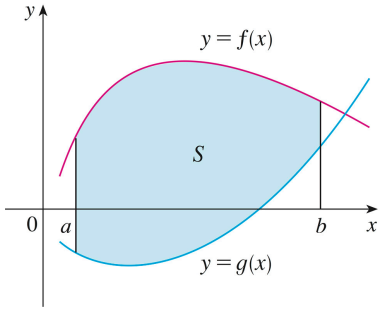
Made with Goodnotes

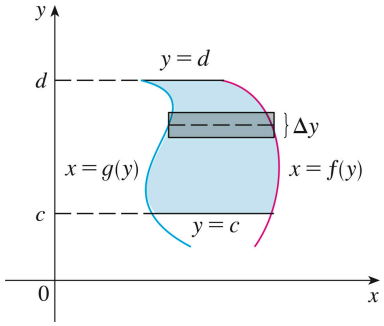
6 Applications of Integration

6.1 Areas between Curves

In this section we use integrals to find areas of regions that lie between the graphs of two functions.

There are 2 main cases:

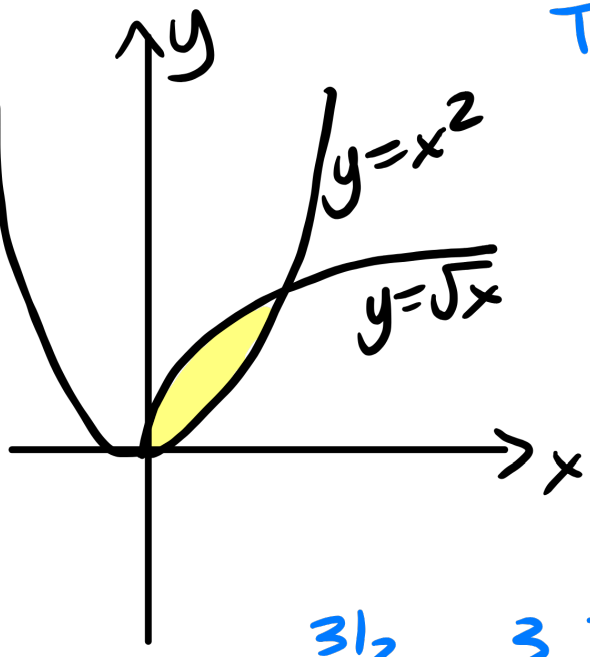




In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ for $f(x) \geq g(x)$. The area is given by $\boxed{A = \int_a^b [f(x) - g(x)]\,dx}$

In the second case we want to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ for $f(y) \geq g(y)$. The area is given by $\boxed{A = \int_c^d [f(y) - g(y)]\,dy}$

Example. Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.



To find limits of integration:

$x^2 = \sqrt{x} \Rightarrow x^2 - \sqrt{x} = 0$

$x^{1/2}(x^{3/2} - 1) = 0$

$x = 0, \; x = 1$

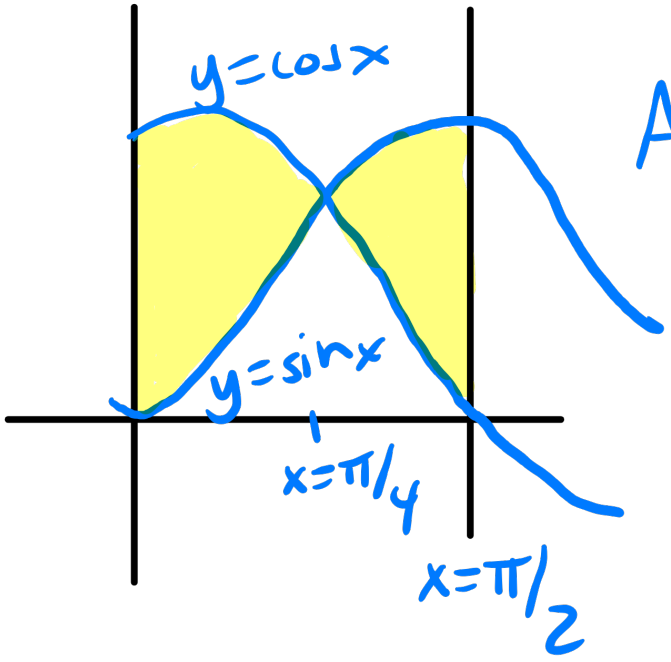
$A = \int_0^1 \sqrt{x} - x^2\,dx$

$= \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big]_0^1 = \frac{2}{3} - \frac{1}{3} - (0 - 0)$

$= \boxed{\frac{1}{3}}$

Note: Unlike the area under a curve that we looked at in the previous chapter, the area between two curves will *always* be positive.

Example. Determine the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = \dfrac{\pi}{2}$, and the y-axis.



$$A = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$= \sin x + \cos x \Big]_0^{\pi/4} + \Big[-\cos x - \sin x \Big]_{\pi/4}^{\pi/2}$$

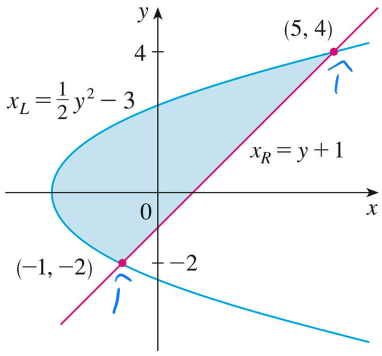
$$= \sin \pi/4 + \cos \pi/4 - \sin 0 - \cos 0 - \cos \pi/2 - \sin \pi/2$$

$$+ \cos \pi/4 + \sin \pi/4 = 4 \cdot \frac{\sqrt{2}}{2} - 2 = \boxed{2\sqrt{2} - 2}$$

Example. Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$, $x = y + 1$.

→ transformation of $x = y^2$

$y = x - 1$



$\frac{1}{2}y^2 - 3 = y + 1$

$\frac{1}{2}y^2 - y - 4 = 0$

$y^2 - 2y - 8 = 0$

$(y-4)(y+2) = 0$

$\Rightarrow y = 4, y = -2$

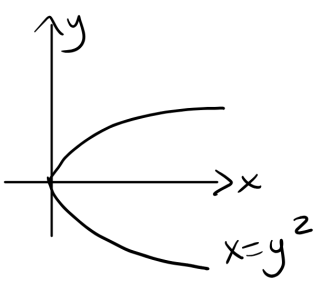
$$A = \int_{-2}^{4} y + 1 - \left(\frac{1}{2}y^2 - 3\right) dy = \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy$$

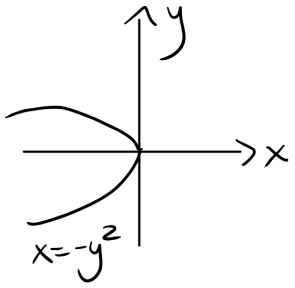
$$= -\frac{1}{2}\cdot\frac{y^3}{3} + \frac{y^2}{2} + 4y \Big]_{-2}^{4} = -\frac{1}{2}\cdot\frac{4^3}{4} + \frac{16}{2} + 16 - \left(-\frac{1}{2}\frac{(-8)}{3} + 2 - 8\right)$$

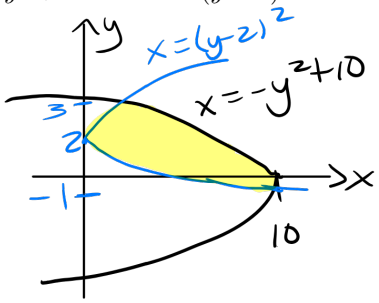
$$= -8 + 8 + 16 - \frac{4}{3} + 6 = 22 - \frac{4}{3} = \frac{66-4}{3} = \boxed{\frac{62}{3}}$$

$x = f(y)$

Example. Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.







$-y^2+10=(y-2)^2$

$-y^2+10=y^2-4y+4$

$0=2y^2-4y-6$

$0=2(y^2-2y-3)$

$0=2(y-3)(y+1)$

$y=3,\ y=-1$

$$A=\int_{-1}^{3} -y^2+10-(y-2)^2\,dy$$

$$=\int_{-1}^{3} -y^2+10-(y^2-4y+4)\,dy$$

$$=\int_{-1}^{3} -2y^2+4y+6\,dy$$

$$=-\frac{2y^3}{3}+2y^2+6y\Big]_{-1}^{3}$$

$$=-2\cdot\frac{3^3}{3}+2\cdot 3^2+6\cdot 3-\left[-\frac{2\cdot(-1)^3}{3}+2(-1)^2-6\right]$$

$$=-18+18+18-\frac{2}{3}+4=22-\frac{2}{3}=\boxed{\frac{64}{3}}$$

Example. Determine the area of the region enclosed by $x = y^2 - 4y$, $x = 2y - y^2$.

Complete the square!

$x = y^2 - 4y + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2$

$(y-2)^2 - 4$

shift $x=y^2$ 2 units up, 4 units to the left

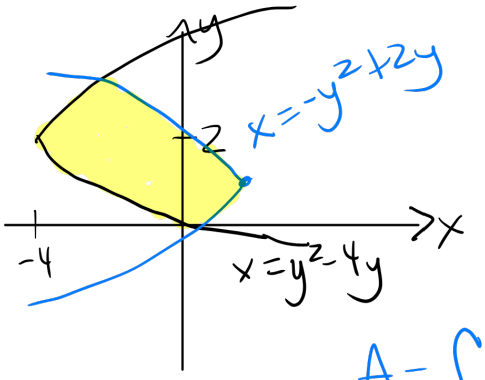
$x = -y^2 + 2y = -\left(y^2 - 2y + \left(\frac{-2}{2}\right)^2 - \left(\frac{-2}{2}\right)^2\right)$

$-\left[(y-1)^2 - 1\right]$

$= -(y-1)^2 + 1$

flip $x=y^2$ across y-axis

shift 1 unit ~~down~~ up, 1 unit to the right



$y^2 - 4y = 2y - y^2$

$2y^2 - 6y = 0$

$2y(y-3) = 0$ $\quad y = 0, y = 3$

$$A = \int_0^3 -y^2 + 2y - (y^2 - 4y)\,dy$$

$$= \int_0^3 -2y^2 + 6y\,dy = -2\frac{y^3}{3} + 3y^2 \Big]_0^3$$

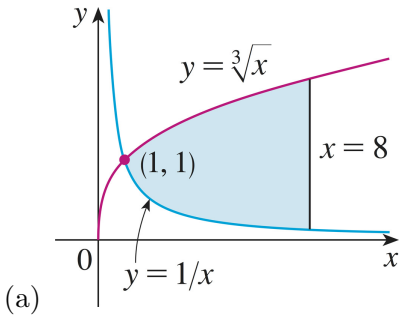
$$= -2 \cdot 9 + 27 - (0+0) = \boxed{9}$$

Homework 6.1 Solutions: Areas between curves

Prof. Leyda Almodóvar

Name: _____

1. Find the area of the shaded region.

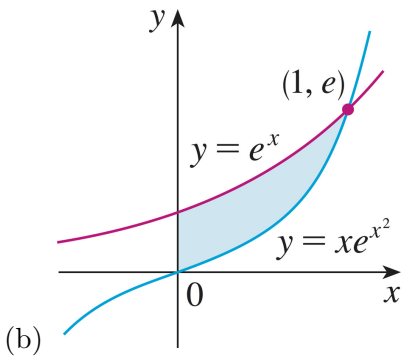


(a)

The curves intersect at the point $(1, 1)$. Then,

$$A = \int_1^8 \sqrt[3]{x} - \tfrac{1}{x}\,dx = \left[\tfrac{3}{4}x^{\frac{4}{3}} - \ln x\right]_1^8 = \tfrac{3}{4}\cdot 8^{\frac{4}{3}} - \ln 8 - \left(\tfrac{3}{4} - \ln 1\right)$$

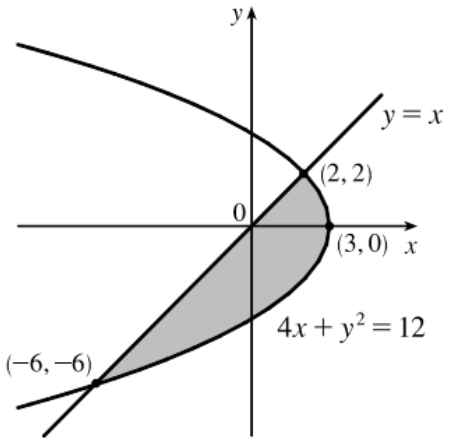
$$= \tfrac{3}{4}(16) - \ln 8 - \tfrac{3}{4} = 12 - \ln 8 - \tfrac{3}{4} = \boxed{\frac{45}{4} - \ln 8}$$



(b)

The curves intersect at the point $(1, e)$. Then,

$$A = \int_0^1 e^x - xe^{x^2}\,dx = \int_0^1 e^x\,dx - \int_0^1 xe^{x^2}\,dx = \Big[e^x\Big]_0^1 - \tfrac{1}{2}\int_0^1 e^u\,du$$
$$= e^1 - e^0 - \tfrac{1}{2}\Big[e^u\Big]_0^1 = e - 1 - \tfrac{1}{2}(e^1 - e^0) = \boxed{\frac{1}{2}e - \frac{1}{2}}$$



(c)

The curves intersect at the points $(-6,-6)$ and $(2,2)$. Note that the curve $4x+y^2=12$ is to the right of the curve $y=x$ on the interval $-6 \le y \le 2$. The equation $4x+y^2=12$ is equivalent to $x = 3 - \frac{y^2}{4}$. Then,

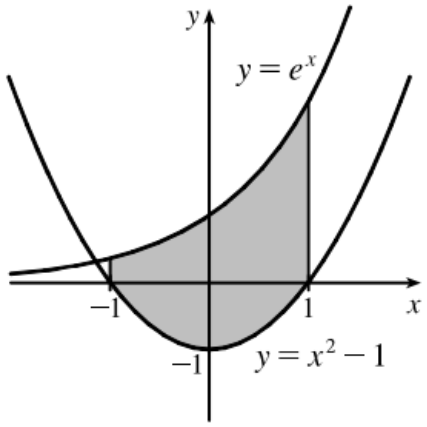
$$A = \int_{-6}^{2} 3 - \tfrac{y^2}{4} - y \, dy = \left[3y - \tfrac{y^3}{12} - \tfrac{y^2}{2} \right]_{-6}^{2}$$

$$= 3(2) - \tfrac{8}{12} - \tfrac{4}{2} - \left(3(-6) - \tfrac{(-6)^3}{12} - \tfrac{36}{2} \right) = 6 - \tfrac{2}{3} - 2 + 18 - 18 + 18 = 22 - \tfrac{2}{3} = \boxed{\frac{64}{3}}$$

Alternatively, we could notice that $y=x$ is on top of $4x+y=12$ when $-6 \le x \le 2$ and then from $2 \le x \le 3$, the curve $4x+y^2=12$ is on top of itself. This would require us to find two separate integrals and to solve the equation $4x+y^2=12$ for y. Then, $A(x) = \int_{-6}^{2} x - (-\sqrt{12-4x}) \, dx + \int_{2}^{3} \sqrt{12-4x} - (-\sqrt{12-4x}) \, dx$.

2. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Then find the area of the region.

   (a) $y = e^x$, $y = x^2 - 1$, $x = -1$, $x = 1$.

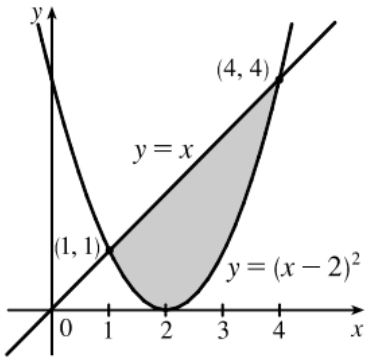


Note that the curve $y = e^x$ is on top of the curve $y = x^2 - 1$ on the interval $-1 \le x \le 1$. Then,

$$A = \int_{-1}^{1} e^x - (x^2 - 1)\,dx = \left[e^x - \tfrac{x^3}{3} + x\right]_{-1}^{1}$$

$$= e^1 - \tfrac{1}{3} + 1 - \left(e^{-1} + \tfrac{1}{3} - 1\right) = e - \tfrac{1}{3} + 1 - \tfrac{1}{e} - \tfrac{1}{3} + 1 = \boxed{e + \frac{4}{3} - \frac{1}{e}}$$

(b) $y = (x-2)^2$, $y = x$



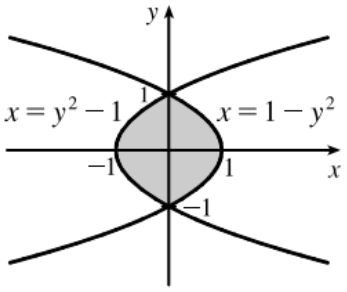
First we find the points of intersection between the curves $y = (x-2)^2$ and $y = x$:

$$\begin{aligned}(x-2)^2 &= x\\ x^2 - 4x + 4 &= x\\ x^2 - 5x + 4 &= 0\\ (x-4)(x-1) &= 0\end{aligned}$$

Hence, $x = 1$ and $x = 4$. Note that the curve $y = x$ is on top of the curve $y = (x-2)^2$ on the interval $1 \le x \le 4$. Then,

$$\begin{aligned}A &= \int_1^4 x - (x^2 - 4x + 4)\,dx = \int_1^4 -x^2 + 5x - 4\,dx = \left[-\tfrac{x^3}{3} + \tfrac{5x^2}{2} - 4x\right]_1^4\\ &= -\tfrac{64}{3} + 5(8) - 4(4) - \left(-\tfrac{1}{3} + \tfrac{5}{2} - 4\right) = -\tfrac{64}{3} + 40 - 16 + \tfrac{1}{3} - \tfrac{5}{2} + 4\\ &= 28 - 21 - \tfrac{5}{2} = 7 - \tfrac{5}{2} = \boxed{\frac{9}{2}}\end{aligned}$$

(c) $x = 1 - y^2$, $x = y^2 - 1$



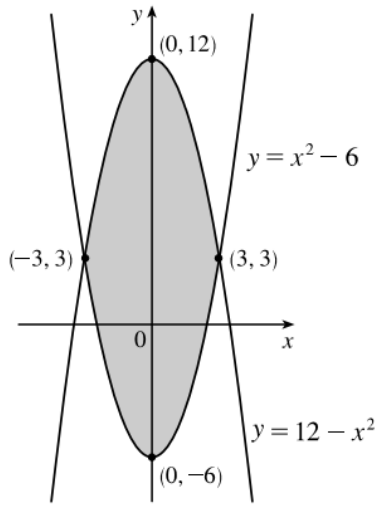
First we find the points of intersection between the curves $x = 1 - y^2$ and $x = y^2 - 1$:

$$\begin{aligned} 1 - y^2 &= y^2 - 1 \\ 0 &= 2y^2 - 2 \\ 0 &= 2(y^2 - 1) \\ 0 &= 2(y + 1)(y - 1) \end{aligned}$$

Hence, $y = -1$ and $y = 1$. Note that the curve $x = 1 - y^2$ is to the right of the curve $x = y^2 - 1$ on the interval $-1 \leq y \leq 1$. Then,

$$\begin{aligned} \text{A} &= \int_{-1}^{1} 1 - y^2 - (y^2 - 1)\,\mathrm{d}y = \int_{-1}^{1} 2 - 2y^2\,\mathrm{d}y = \left[2y - \tfrac{2y^3}{3}\right]_{-1}^{1} \\ &= 2(1) - \tfrac{2}{3} - \left(2(-1) - \tfrac{-2}{3}\right) = 2 - \tfrac{2}{3} + 2 - \tfrac{2}{3} = \boxed{\frac{8}{3}} \end{aligned}$$

(d) $y = 12 - x^2$, $y = x^2 - 6$



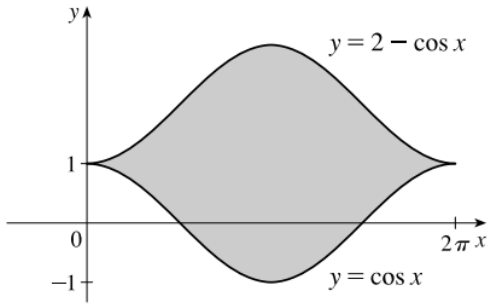
First we find the points of intersection between the curves $y = 12 - x^2$ and $y = x^2 - 6$:

$$\begin{aligned} 12 - x^2 &= x^2 - 6 \\ 0 &= 2x^2 - 18 \\ 0 &= 2(x^2 - 9) \\ 0 &= 2(x + 3)(x - 3) \end{aligned}$$

Hence, $x = -3$ and $x = 3$. Note that the curve $y = 12 - x^2$ is on top of the curve $x^2 - 6$ on the interval $-3 \leq x \leq 3$. Then,

$$A = \int_{-3}^{3} 12 - x^2 - (x^2 - 6)\,\mathrm{d}x = 2\int_{0}^{3} 18 - 2x^2\,\mathrm{d}x \text{ (by symmetry)}$$
$$= \left[18x - \tfrac{2x^3}{3}\right]_0^3 = 2\left(18(3) - \tfrac{54}{3} - 18(0) + \tfrac{0}{3}\right) = 2(54 - 18) = \boxed{72}$$

(e) $y = \cos x$, $y = 2 - \cos x$, $0 \leq x \leq 2\pi$



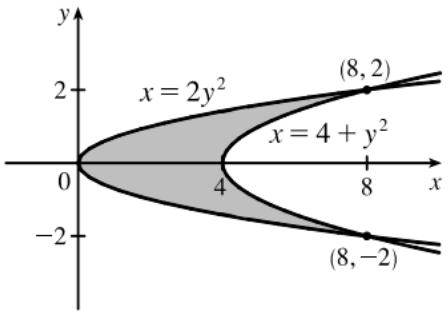
First we find the points of intersection between the curves $y = \cos x$ and $y = 2 - \cos x$:

$$\begin{aligned} \cos x &= 2 - \cos x \\ 2\cos x &= 2 \\ \cos x &= 1 \end{aligned}$$

Hence, $x = 0$ and $x = 2\pi$. Note that the curve $y = 2 - \cos x$ is on top of the curve $y = \cos x$ on the interval $0 \leq x \leq 2\pi$. Then,

$$\begin{aligned} A = \int_0^{2\pi} 2 - \cos x - \cos x \, dx &= \int_0^{2\pi} 2 - 2\cos x \, dx = \Big[2x - 2\sin x\Big]_0^{2\pi} \\ &= 2(2\pi) - 2\sin(2\pi) - (2(0) - 2\sin(0)) = \boxed{4\pi} \end{aligned}$$

(f) $x = 2y^2$, $x = 4 + y^2$



First we find the points of intersection between the curves $x = 2y^2$ and $x = 4+y^2$:

$$\begin{aligned} 2y^2 &= 4 + y^2 \\ y^2 - 4 &= 0 \\ (y+2)(y-2) &= 0 \end{aligned}$$

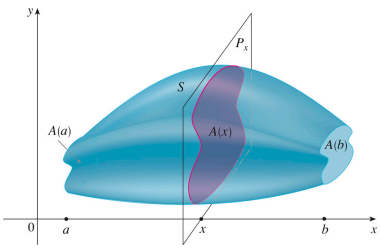
Hence, $y = -2$ and $y = 2$. Note that the curve $x = 4 + y^2$ is to the right of the curve $x = 2y^2$ on the interval $-2 \le y \le 2$. Then,

$$\begin{aligned} A &= \int_{-2}^{2} 4 + y^2 - 2y^2 \, dy = \int_{-2}^{2} 4 - y^2 \, dy = \left[4y - \tfrac{y^3}{3}\right]_{-2}^{2} \\ &= 4(2) - \tfrac{8}{3} - \left(4(-2) - \tfrac{-8}{3}\right) = 8 - \tfrac{8}{3} + 8 - \tfrac{8}{3} = \boxed{\frac{32}{3}} \end{aligned}$$

6.2 Volumes

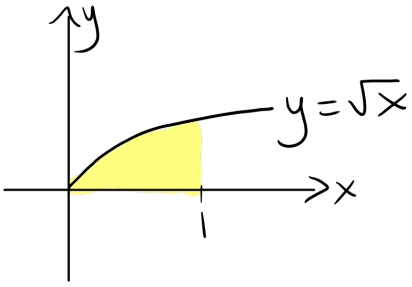
In this section we will find the volume of a solid of revolution by "cutting" the solid into pieces and approximating each piece by a disk or annular ring.

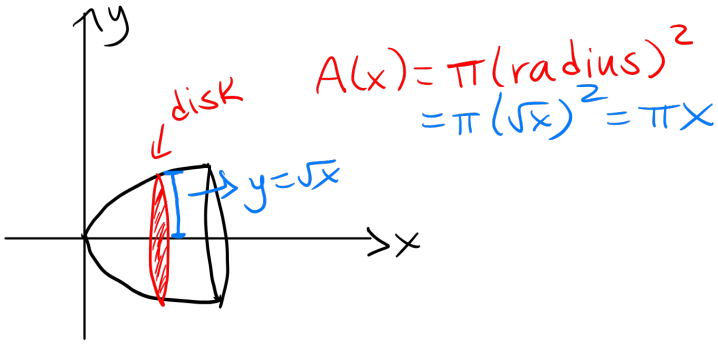
- We rotate a curve $y = f(x)$ on $[a, b]$ about a given axis to get the surface of the solid.
- In order to find the volume of the solid, we need to find the *cross-sectional area* of the solid.
- In this section, we will get a cross-section by cutting the solid perpendicular to the axis of rotation.
- The cross-section will be either a solid disk if the object is solid or a ring if we've hollowed out a portion of the solid.



- In the case that we get a solid disk the area is $\boxed{A = \pi(\text{radius})^2}$
- In the case that we get a ring (called a *washer*), the area is $\boxed{A = \pi[(\text{outer radius})^2 - (\text{inner radius})^2]}$
- In either case, $\boxed{V = \int_a^b A(x)\,dx}$ or $\boxed{V = \int_c^d A(y)\,dy}$

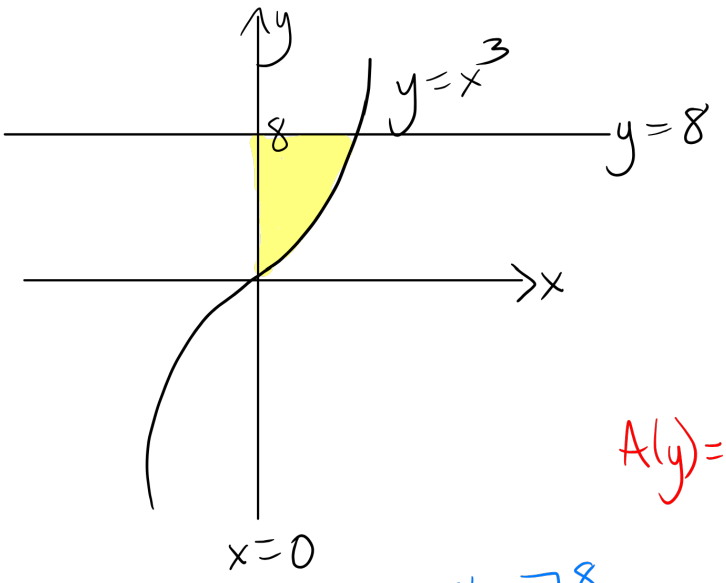
Example. Determine the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

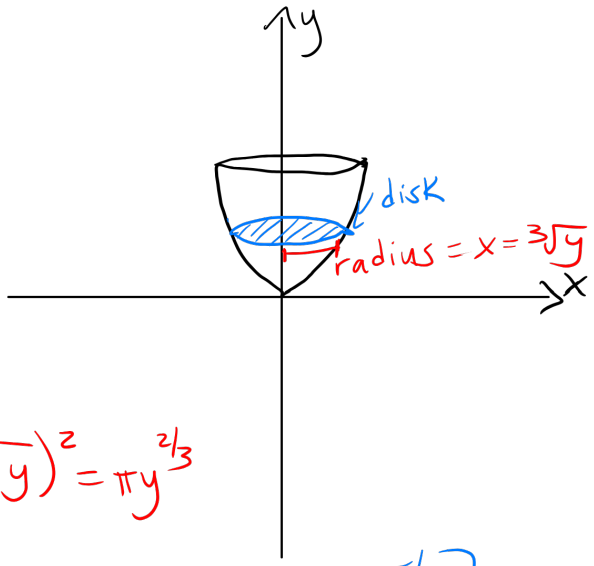




$$V = \int_0^1 \pi x\,dx = \pi\frac{x^2}{2}\Big]_0^1 = \pi\cdot\left(\frac{1}{2} - 0\right) = \boxed{\frac{\pi}{2}}$$

Example. Determine the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.



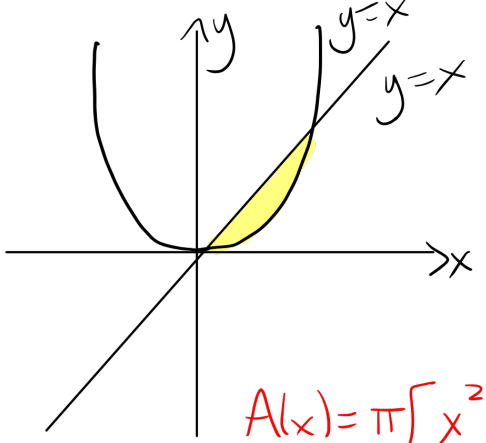


$$A(y) = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

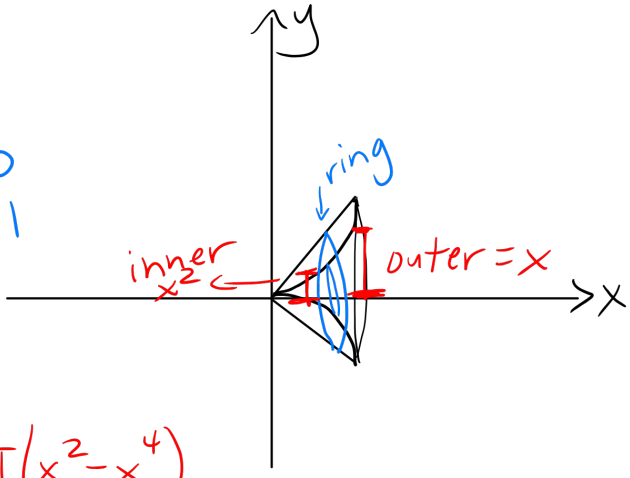
$$V = \int_0^8 \pi y^{2/3}\,dy = \pi \frac{y^{5/3}}{5/3}\Big]_0^8 = \frac{3\pi}{5} y^{5/3}\Big]_0^8 = \frac{3\pi}{5}\left[8^{5/3} - 0^{5/3}\right]$$

$$= \frac{3\pi}{5}(32) = \boxed{\frac{96\pi}{5}}$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the x-axis.



$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0, x = 1$$

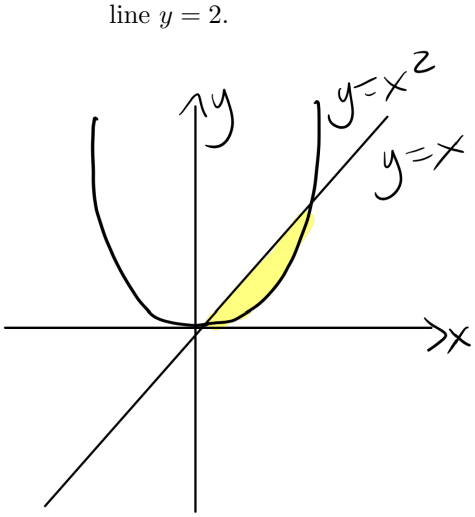


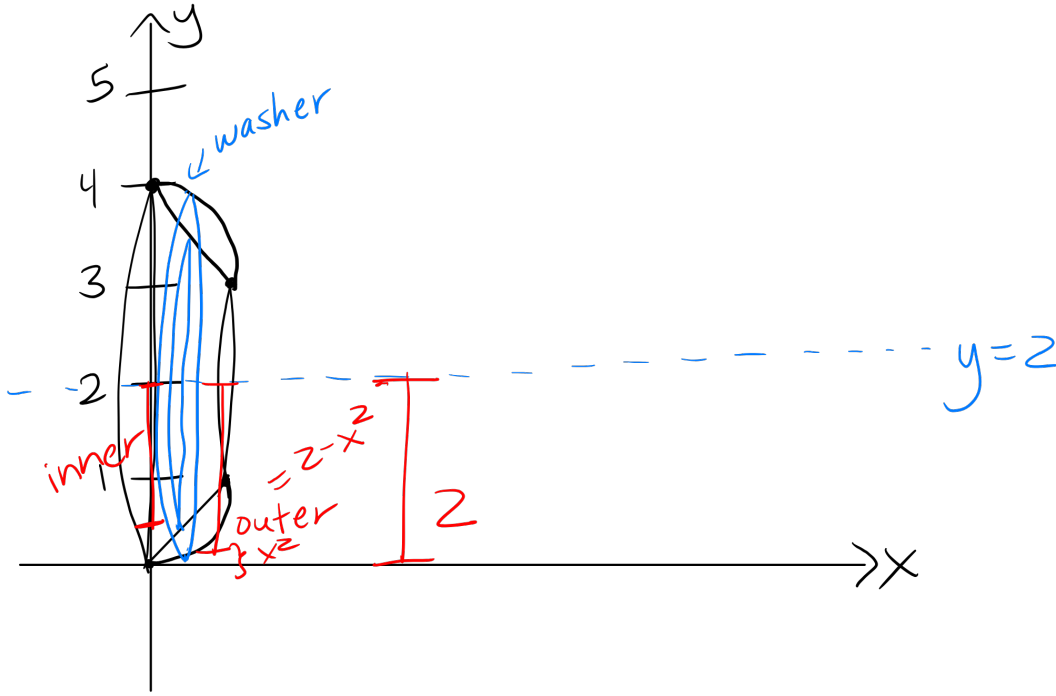
$$A(x) = \pi\left[x^2 - (x^2)^2\right] = \pi(x^2 - x^4)$$

$$V = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left[\frac{1}{3} - \frac{1}{5} - (0-0)\right]$$

$$= \boxed{\frac{2\pi}{15}}$$

Example. Find the volume of the solid obtained by rotating the region in the previous example about the line $y = 2$.





outer radius $= 2-x^2$

inner radius $= 2-x$

$$A(x)=\pi\left[(2-x^2)^2-(2-x)^2\right]=\pi\left[4-4x^2+x^4-(4-4x+x^2)\right]$$

$$=\pi\left(x^4-5x^2+4x\right)$$

$$V=\int_0^1 \pi\left(x^4-5x^2+4x\right)dx=\pi\left[\frac{x^5}{5}-\frac{5x^3}{3}+2x^2\right]_0^1$$

$$=\pi\left[\frac{1}{5}-\frac{5}{3}+2-(0-0+0)\right]=\pi\left(\frac{3-25+30}{15}\right)=\boxed{\frac{8\pi}{15}}$$

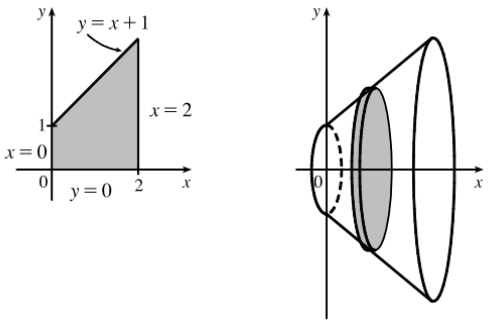
Homework 6.2 Solutions: Volumes

Prof. Leyda Almodóvar

Name: _____

1. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

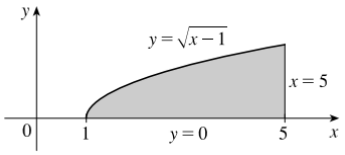
   (a) $y = x + 1$, $y = 0$, $x = 0$, $x = 2$; about the x-axis

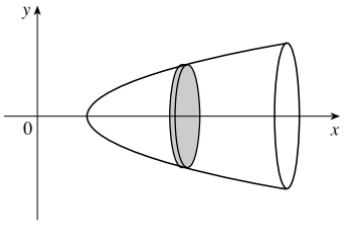


A cross-section is a disk with radius $x + 1$, so its area is $A(x) = \pi(x + 1)^2 = \pi(x^2 + 2x + 1)$. Then, the volume is given by

$$V = \int_0^2 A(x)\,dx = \int_0^2 \pi(x^2 + 2x + 1)\,dx = \pi\left[\tfrac{x^3}{3} + x^2 + x\right]_0^2$$
$$= \pi\left(\tfrac{8}{3} + 4 + 2 - \tfrac{0}{3} - 0^2 - 0\right) = \boxed{\frac{26\pi}{3}}$$

(b) $y = \sqrt{x-1}$, $y = 0$, $x = 5$; about the x-axis

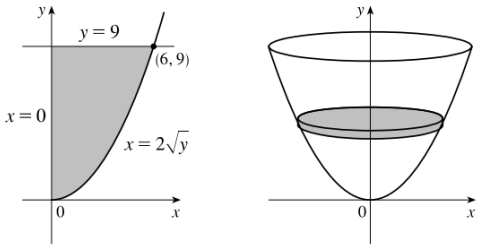




A cross-section is a disk with radius $\sqrt{x-1}$, so its area is $A(x) = \pi(\sqrt{x-1})^2 = \pi(x-1)$. Then, the volume is given by

$$V = \int_1^5 A(x)\,\mathrm{d}x = \int_1^5 \pi(x-1)\,\mathrm{d}x = \pi\left[\frac{x^2}{2} - x\right]_1^5$$
$$= \pi\left[\frac{25}{2} - 5 - \left(\frac{1}{2} - 1\right)\right] = \pi(12 - 5 + 1) = \boxed{8\pi}$$

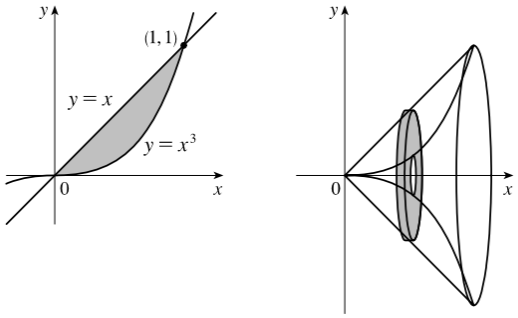
(c) $x = 2\sqrt{y}$, $x = 0$, $y = 9$; about the y-axis



A cross-section is a disk with radius $2\sqrt{y}$, so its area is $A(y) = \pi(2\sqrt{y})^2 = 4\pi y$. Then, the volume is given by

$$V = \int_0^9 A(y)\,dy = \int_0^9 4\pi y\,dy = 4\pi\left[\frac{y^2}{2}\right]_0^9$$
$$= 4\pi\left(\frac{81}{2} - \frac{0}{2}\right) = \boxed{162\pi}$$

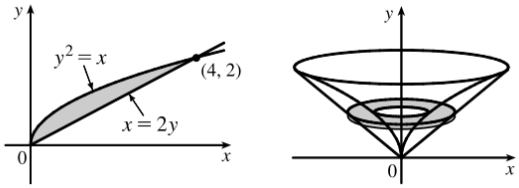
(d) $y = x^3$, $y = x$, $x \geq 0$; about the x-axis



A cross-section is a washer with inner radius x^3 and outer radius x, so its area is $A(x) = \pi[x^2 - (x^3)^2] = \pi(x^2 - x^6)$. The curves intersect at the points $x = 0$ and $x = 1$ (found by setting $x^3 = x$). Then, the volume is given by

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^6)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^7}{7}\right]_0^1$$
$$= \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \boxed{\frac{4\pi}{21}}$$

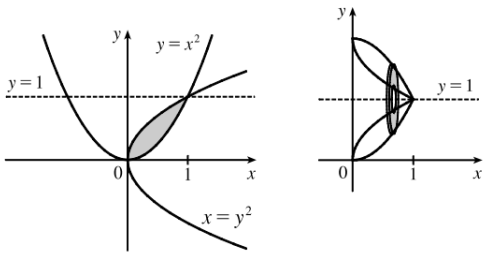
(e) $y^2 = x$, $x = 2y$; about the y-axis



A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is $A(y) = \pi[(2y)^2 - (y^2)^2] = \pi(4y^2 - y^4)$. The curves intersect at the points $y = 0$ and $y = 2$ (found by setting $y^2 = 2y$). Then, the volume is given by

$$V = \int_0^2 A(y)\,dy = \int_0^2 \pi(4y^2 - y^4)\,dy = \pi\left[\frac{4y^3}{3} - \frac{y^5}{5}\right]_0^2$$
$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi\left(\frac{1}{3} - \frac{1}{5}\right) = 32\pi\left(\frac{5-3}{15}\right) = \boxed{\frac{64\pi}{15}}$$

(f) $y = x^2$, $x = y^2$; about $y = 1$



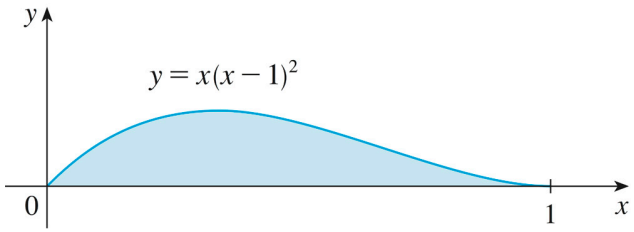
A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x^2$, so its area is $A(x) = \pi[(1 - x^2)^2 - (1 - \sqrt{x})^2] = \pi(x^4 - 2x^2 - x + 2\sqrt{x})$. The curves intersect at the points $x = 0$ and $x = 1$ (found by setting $x^2 = \sqrt{x}$). Then, the volume is given by

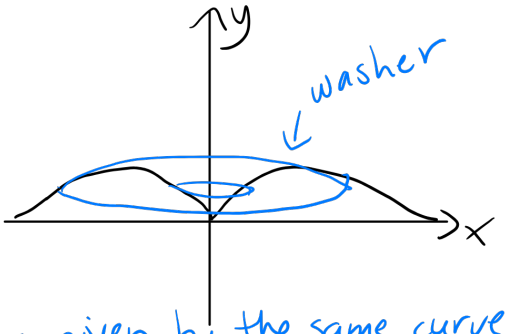
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^4 - 2x^2 - x + 2\sqrt{x})\,dx = \pi\left[\tfrac{x^5}{5} - \tfrac{2x^3}{3} - \tfrac{x^2}{2} + \tfrac{4}{3}x^{\frac{3}{2}}\right]_0^1$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{2}{3} - \tfrac{1}{2} + \tfrac{4}{3}\right) = \boxed{\frac{11\pi}{30}}$$

6.3 Volumes by Cylindrical Shells

In this section we will use the method of cylindrical shells to compute volumes that result in complicated integrals using disks or washers.

Example. Determine the volume of the solid obtained by rotating the region bounded by $y = x(x-1)^2$ and the x-axis about the y-axis.

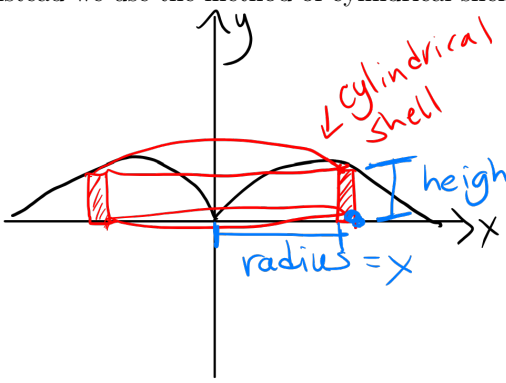




Issues:

- The inner radius and the outer radius are given by the same curve.
- We would need to write $x = f(y)$, not easy for a cubic function.

Instead we use the method of cylindrical shells:



- Draw a strip parallel to the axis of rotation.
- Rotate the strip around the solid

The area of a cylindrical shell is given by $\boxed{A = 2\pi(\text{radius})(\text{height})}$

The volume is still given by $V = \int_a^b A(x)\,dx$ or $V = \int_c^d A(y)\,dy$.

$$A(x) = 2\pi x \cdot x(x-1)^2 = 2\pi x^2(x^2-2x+1) = 2\pi(x^4-2x^3+x^2)$$

$$V = \int_0^1 2\pi(x^4-2x^3+x^2)\,dx = 2\pi\left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3}\right]_0^1$$

$$= 2\pi\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3} - (0-0+0)\right] = 2\pi\left(\frac{6-15+10}{30}\right) = \boxed{\frac{\pi}{15}}$$

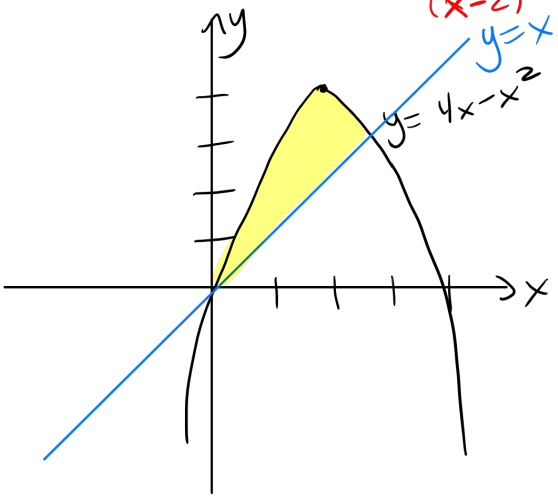
Differences between this method and the previous methods

- Rotation about a vertical axis will give an area that is a function of x and rotation about a horizontal axis will give an area that is a function of y. (Opposite of previous section)
- We don't take the complete range of x or y for the limits of integration. Instead we take a range of x or y that will cover one side of the solid.

Example. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = x$ about the y-axis.

$y = -x^2+4x = -\left(x^2-4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2\right) = -(x-2)^2+4$

$(x-2)^2$

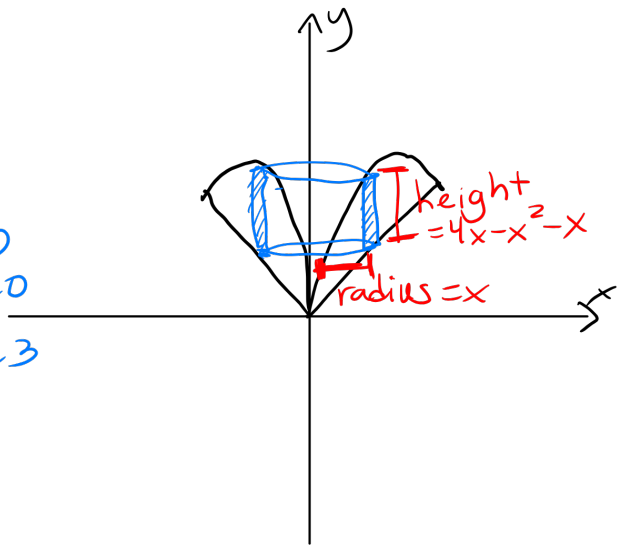


$x = 4x - x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, x = 3$

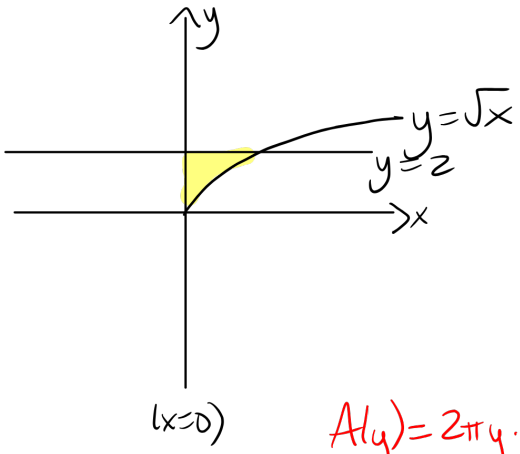


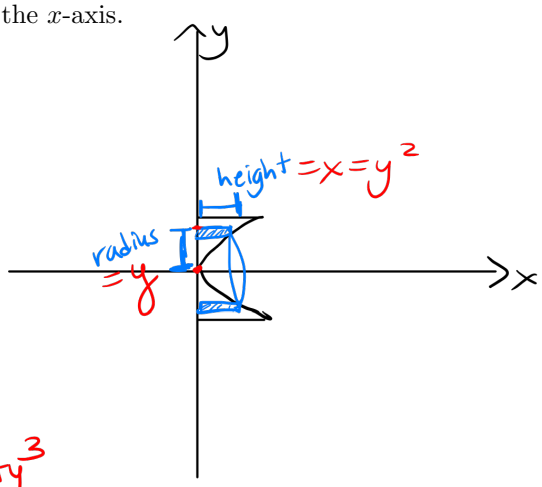
$A(x) = 2\pi x \cdot (3x - x^2) = 2\pi(3x^2 - x^3)$

$V = \int_0^3 2\pi(3x^2 - x^3)\,dx = 2\pi\left[x^3 - \frac{x^4}{4}\right]_0^3 = 2\pi\left[3^3 - \frac{3^4}{4} - (0-0)\right]$

$2\pi\left[\frac{108-81}{4}\right] = \boxed{\frac{27\pi}{2}}$

Example. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 0$, and $y = 2$ about the x-axis.

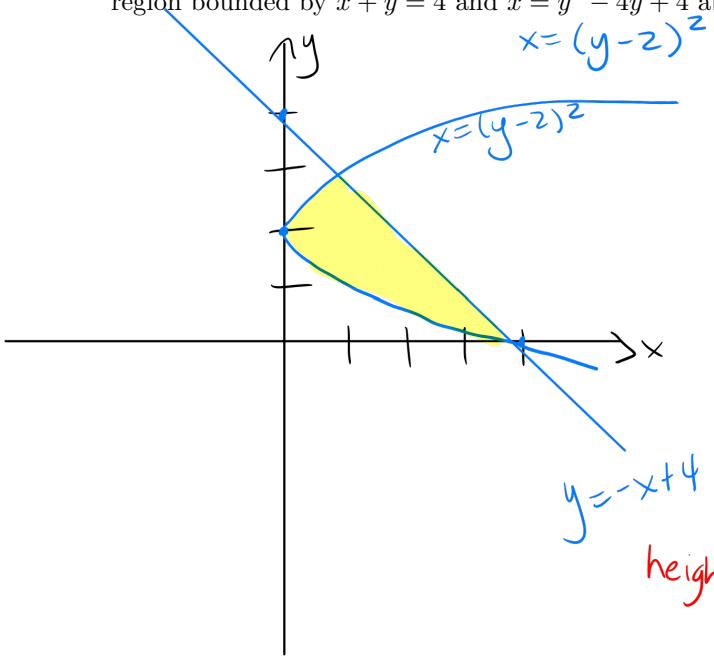


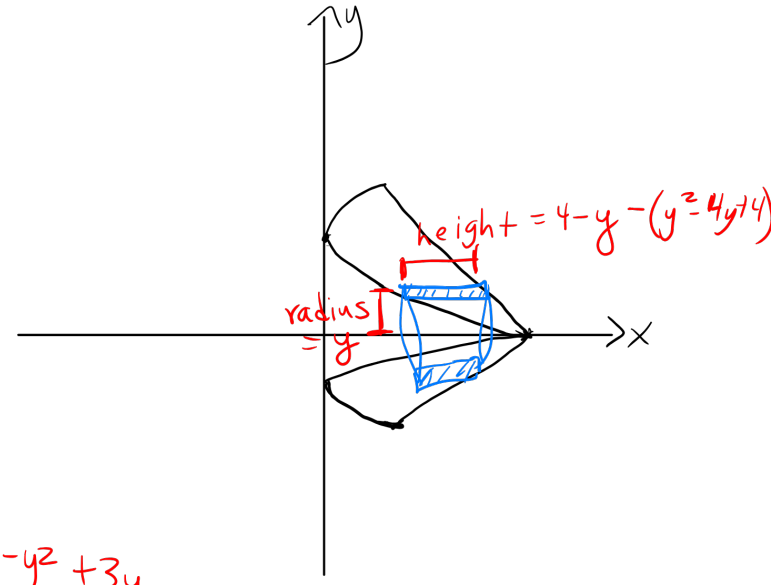


$A(y) = 2\pi y \cdot y^2 = 2\pi y^3$

$$V = \int_0^2 2\pi y^3 \, dy = 2\pi \frac{y^4}{4} \Big]_0^2 = 2\pi \cdot \left[\frac{16}{4} - 0\right] = \boxed{8\pi}$$

Example. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $x + y = 4$ and $x = y^2 - 4y + 4$ about the x-axis.





$y = 4 - x$

height $= -y^2 + 3y$

$A(y) = 2\pi(y)(-y^2+3y) = 2\pi(-y^3+3y^2)$

$y^2 - 4y + 4 = 4 - y$

$y^2 - 3y = 0$

$y(y-3) = 0$

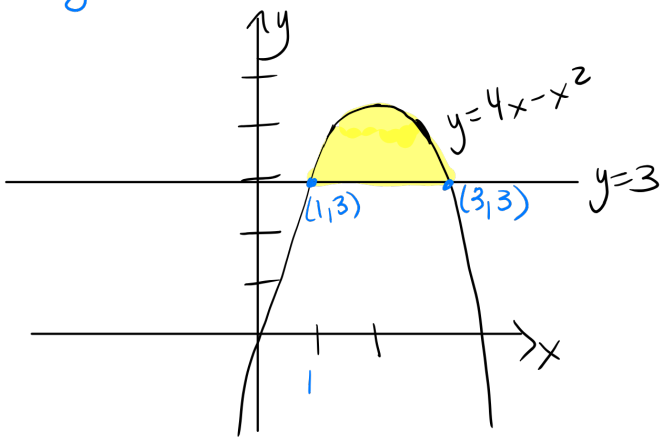
$y = 0, y = 3$

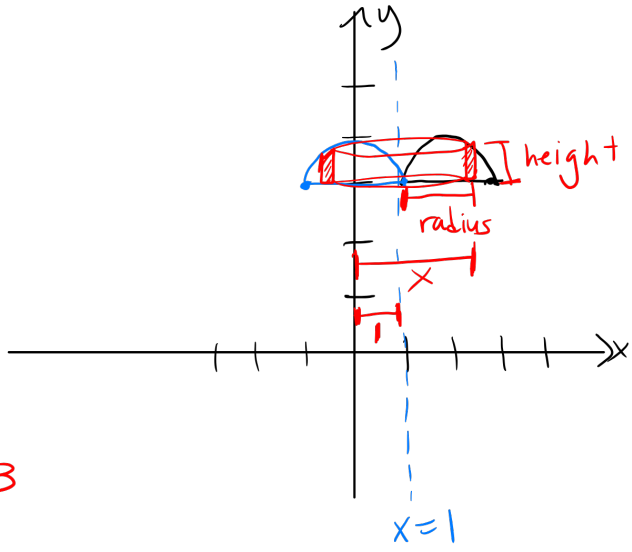
$$V = \int_0^3 2\pi(-y^3+3y^2)\,dy$$

$$= 2\pi\left[-\frac{y^4}{4} + y^3\right]_0^3 = 2\pi\left[-\frac{81}{4} + 27\right] = 2\pi\left[27/4\right] = \boxed{\frac{27\pi}{2}}$$

Example. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about $x = 1$.

$y = 4x - x^2 = -(x-2)^2 + 4$





$4x - x^2 = 3$

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x = 3, \; x = 1$

$\text{height} = 4x - x^2 - 3$

$\text{radius} = x - 1$

$A(x) = 2\pi(x-1)(4x - x^2 - 3)$

$= 2\pi(4x^2 - x^3 - 3x - 4x + x^2 + 3)$

$= 2\pi(-x^3 + 5x^2 - 7x + 3)$

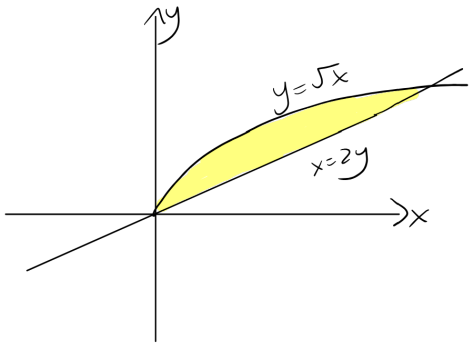
$$V = \int_1^3 2\pi(-x^3 + 5x^2 - 7x + 3)\,dx = 2\pi\left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{7x^2}{2} + 3x\right]_1^3$$

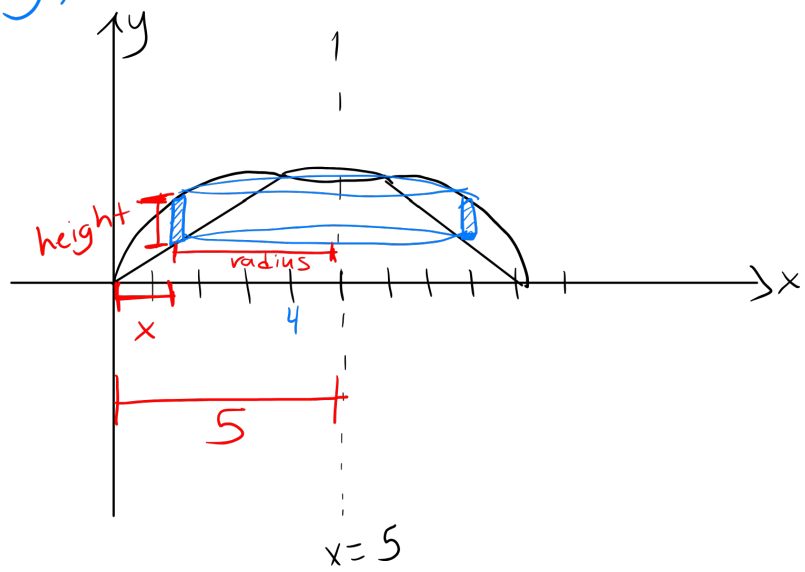
$$= 2\pi\left[\frac{-3^4}{4} + 5 \cdot 9 - \frac{7 \cdot 9}{2} + 9 - \left(\frac{-1}{4} + \frac{5}{3} - \frac{7}{2} + 3\right)\right]$$

$$= \cdots = \boxed{\frac{8\pi}{3}}$$

Example. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about $x = 1$.

$y=\sqrt{x}$, $x=2y$, about $x=5$





$\sqrt{x} = \frac{x}{2}$

$\sqrt{x} - \frac{x}{2} = 0$

$\sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right) = 0$

$x=0$ $\quad 1 = \frac{\sqrt{x}}{2} \Rightarrow 2 = \sqrt{x} \Rightarrow 4 = x$

$\text{height} = \sqrt{x} - \frac{x}{2}$

$\text{radius} = 5 - x$

$$A(x) = 2\pi(5-x)\left(\sqrt{x} - \frac{x}{2}\right) = 2\pi\left(5\sqrt{x} - \frac{5}{2}x - x^{3/2} + \frac{x^2}{2}\right)$$

$$V = \int_0^4 2\pi\left(5\sqrt{x} - \frac{5}{2}x - x^{3/2} + \frac{x^2}{2}\right)dx$$

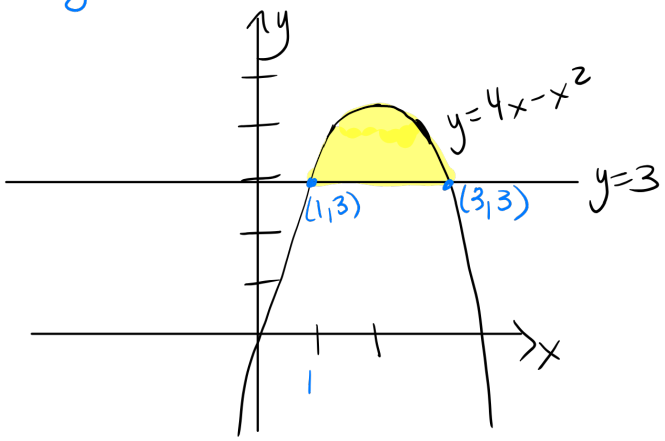
$$= 2\pi\left[5\cdot\frac{2}{3}x^{3/2} - \frac{5}{2}\cdot\frac{x^2}{2} - \frac{2}{5}x^{5/2} + \frac{x^3}{6}\right]_0^4$$

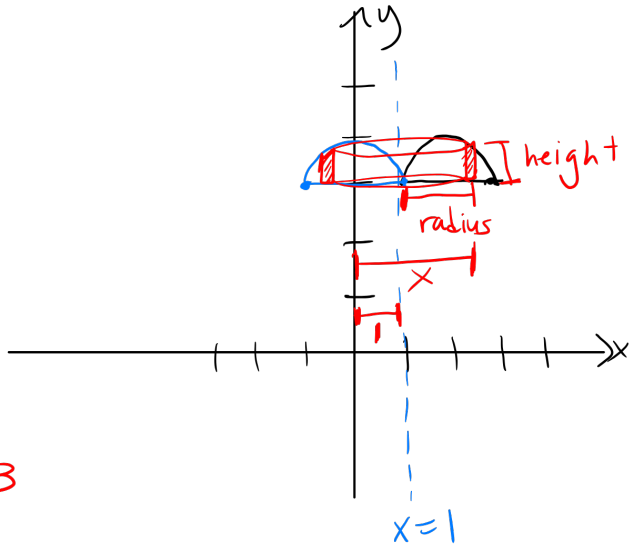
$$= 2\pi\left[\frac{10}{3}\cdot 4^{3/2} - \frac{5}{4}\cdot 16 - \frac{2}{5}\cdot 4^{5/2} + \frac{4^3}{6} - (0-0-0+0)\right]$$

$$= \cdots = \boxed{\frac{136\pi}{15}}$$

Example. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about $x = 1$.

$y = 4x - x^2 = -(x-2)^2 + 4$





$4x - x^2 = 3$

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x = 3, \; x = 1$

$\text{height} = 4x - x^2 - 3$

$\text{radius} = x - 1$

$A(x) = 2\pi(x-1)(4x - x^2 - 3)$

$= 2\pi(4x^2 - x^3 - 3x - 4x + x^2 + 3)$

$= 2\pi(-x^3 + 5x^2 - 7x + 3)$

$$V = \int_1^3 2\pi(-x^3 + 5x^2 - 7x + 3)\,dx = 2\pi\left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{7x^2}{2} + 3x\right]_1^3$$

$$= 2\pi\left[\frac{-3^4}{4} + 5 \cdot 9 - \frac{7 \cdot 9}{2} + 9 - \left(\frac{-1}{4} + \frac{5}{3} - \frac{7}{2} + 3\right)\right]$$

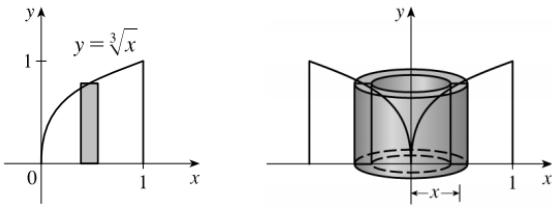
$$= \cdots = \boxed{\frac{8\pi}{3}}$$

Homework 6.3: Volumes by cylindrical shells

Prof. Leyda Almodóvar

Name: _____

1. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = \sqrt[3]{x}$, $y = 0$, $x = 1$ about the y-axis.

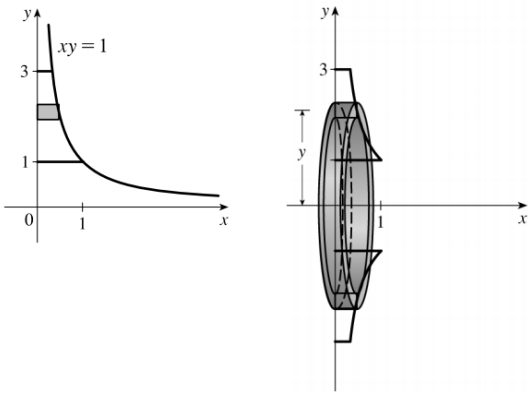


Note that the radius of a cylindrical shell is x while the height is $\sqrt[3]{x}$.
Then, $A(x) = 2\pi x(\sqrt[3]{x}) = 2\pi x^{\frac{4}{3}}$. Hence,

$$V = \int_0^1 2\pi x^{\frac{4}{3}}\,dx = 2\pi\left[\tfrac{3}{7}x^{\frac{7}{3}}\right]_0^1 = 2\pi\left(\tfrac{3}{7}\right) = \boxed{\frac{6\pi}{7}}$$

2. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

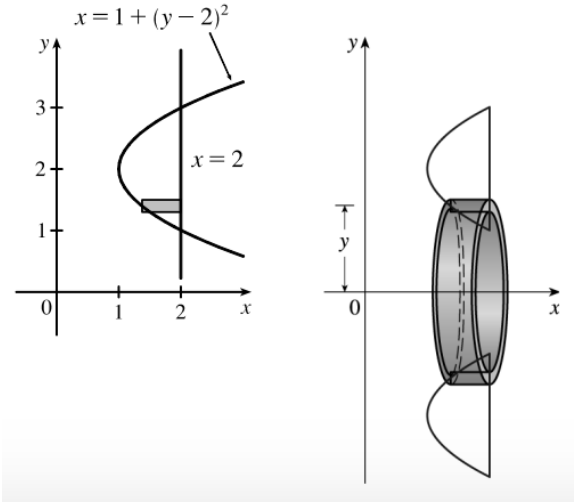
   (a) $xy = 1$, $x = 0$, $y = 1$, $y = 3$



Note that the radius of a cylindrical shell is y while the height is $x = \frac{1}{y}$. Then, $A(y) = 2\pi \frac{1}{y}(y) = 2\pi$. Hence,

$$V = \int_1^3 2\pi \, dy = 2\pi \Big[y \Big]_1^3 = 2\pi(3 - 1) = \boxed{4\pi}$$

(b) $x = 1 + (y-2)^2$, $x = 2$

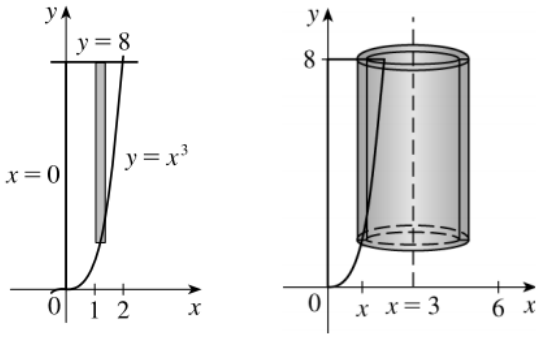


Note that the radius of a cylindrical shell is y while the height is $2 - x = 2 - [1 + (y-2)^2] = -y^2 + 4y - 3$.
Then, $A(y) = 2\pi y(-y^2 + 4y - 3) = 2\pi(-y^3 + 4y^2 - 3y)$. Also, the curves $x = 1 + (y-2)^2$ and $x = 2$ intersect when $y = 1$ and $y = 3$. Hence,

$$V = \int_1^3 2\pi(-y^3 + 4y^2 - 3y)\,dy = 2\pi\left[\frac{-y^4}{4} + \frac{4y^3}{3} - \frac{3y^2}{2}\right]_1^3 =$$
$$2\pi\left[\frac{-81}{4} + 36 - \frac{27}{2} - \left(\frac{-1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right] = \boxed{\frac{16\pi}{3}}$$

3. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

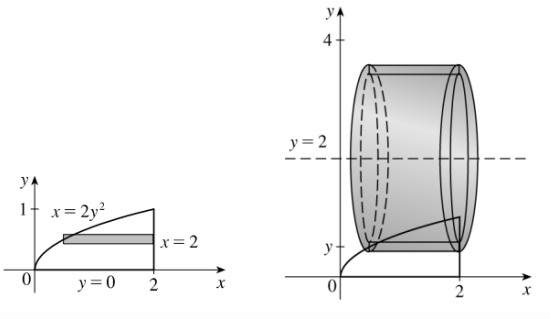
(a) $y = x^3$, $y = 8$, $x = 0$; about $x = 3$



Note that the radius of a cylindrical shell is $3 - x$ while the height is $8 - x^3$. Then, $A(x) = 2\pi(3 - x)(8 - x^3) = 2\pi(24 - 3x^3 - 8x + x^4)$. Also, the curves $y = x^3$ and $y = 8$ intersect when $x = 0$ and $x = 2$. Hence,

$$V = \int_0^2 2\pi(24 - 3x^3 - 8x + x^4)\,dx = 2\pi\left[24x - \tfrac{3x^4}{4} - 4x^2 + \tfrac{x^5}{5}\right]_0^2$$
$$= 2\pi\left(48 - 12 - 16 + \tfrac{32}{5}\right) = \boxed{\frac{264\pi}{5}}$$

(b) $x = 2y^2$, $y \geq 0$, $x = 2$; about $y = 2$



Note that the radius of a cylindrical shell is $2 - y$ while the height is $2 - 2y^2$. Then, $A(y) = 2\pi(2-y)(2-2y^2) = 4\pi(y^3-2y^2-y+2)$. Also, the curves $x = 2y^2$ and $x = 2$ intersect when $y = 1$ (since $y \geq 0$). Hence,

$$V = \int_0^1 4\pi(y^3 - 2y^2 - y + 2)\,dy = 4\pi\left[\tfrac{y^4}{4} - \tfrac{2y^3}{3} - \tfrac{y^2}{2} + 2y\right]_0^1 =$$

$$4\pi\left(\tfrac{1}{4} - \tfrac{2}{3} - \tfrac{1}{2} + 2\right) = \boxed{\frac{13\pi}{3}}$$

7 Methods of Integration

7.1 Integration by Parts

- Every differentiation rule has a corresponding integration rule. For example, the Substitution Rule for integration corresponds to the Chain Rule for differentiation.
- The rule that corresponds to the Product Rule for differentiation is called the rule for **integration by parts**.
- Recall the Product Rule: $[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$.
- Integrate on both sides: $f(x)g(x) = \int [f(x)g'(x) + g(x)f'(x)]\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$.
- Rearrange the equation: $\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$.
- This equation is easier to remember if we change notation: $u = f(x)$, $v = g(x)$, then $du = f'(x)\,dx$, $dv = g'(x)\,dx$, so

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example. Find $\int x\cos x\,dx$.

Need to pick a function to be "u", the rest is "dv".
"u" will be differentiated while "dv" will be integrated.
Pick "u" such that "du" is simpler than "u"

Option #1: $u = x$, $du = dx$; $dv = \cos x\,dx$, $v = \int \cos x\,dx = \sin x$ ← du got simpler

Option #2: $u = \cos x$, $du = -\sin x\,dx$; $dv = x\,dx$, $v = \int x\,dx = \frac{x^2}{2}$ →

$$\int x\cos x\,dx = \frac{x^2}{2}\cos x + \int \frac{x^2}{2}\sin x\,dx$$

more complicated

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + C$$

$$= \boxed{x\sin x + \cos x + C}$$

Example. Find $\int xe^x \, dx$.

$u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = \int e^x \, dx = e^x$

$$\int xe^x \, dx = xe^x - \int e^x \, dx = \boxed{xe^x - e^x + C}$$

Example. Find $\int x^2 e^x \, dx$

$u = x^2 \qquad dv = e^x \, dx$

$du = 2x \, dx \qquad v = e^x$

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx = x^2 e^x - 2\left[xe^x - e^x\right] + C$$

$$= \boxed{x^2 e^x - 2xe^x + 2e^x + C}$$

$U = x \qquad dV = e^x \, dx$

$dU = dx \qquad V = e^x$

Example. Evaluate $\int e^x \sin x \, dx$.

Note: u could be e^x as well

$u = \sin x \quad dv = e^x dx$

$du = \cos x \, dx \quad v = e^x$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$$

$u = \cos x \quad dV = e^x dx$

$dU = -\sin x \, dx \quad V = e^x$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\Rightarrow 2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \boxed{\frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C}$$

We can also evaluate definite integrals by parts: $\boxed{\int_a^b f(x)g'(x)\,dx = [f(x)(g(x)]_a^b - \int_a^b g(x)f'(x)\,dx}$

Example. Find $\int_1^2 \ln x\,dx$.

Note: Not the integral of a product of functions

$u=\ln x \quad dv=dx$

$du=\frac{1}{x}dx \quad v=\int 1\,dx = x$

$$\int_1^2 \ln x\,dx = [x\ln x]_1^2 - \int_1^2 \frac{1}{x}\cdot x\,dx = [x\ln x]_1^2 - \int_1^2 1\,dx$$

$$= [x\ln x]_1^2 - [x]_1^2 = 2\ln 2 - 2 - (1\cdot\ln 1 - 1) = 2\ln 2 - 2 + 1 = \boxed{2\ln 2 - 1}$$

Note: $\boxed{\int \ln x\,dx = x\ln x - x}$

Example. Calculate $\int_0^1 \tan^{-1} x\,dx$

$u=\tan^{-1}x \quad dv=dx$

$du=\frac{1}{x^2+1}dx \quad v=x$

$$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= 1\cdot\tan^{-1}(1) - 0 - \int_1^2 \frac{1}{2}\cdot\frac{1}{u}\,du$$

$u=x^2+1 \quad x=0,\ u=1$

$du=2x\,dx \quad x=1,\ u=2$

$\frac{1}{2}du=x\,dx$

$$= \pi/4 - \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$$= \pi/4 - \left[\frac{1}{2}\ln|u|\right]_1^2$$

$$= \pi/4 - \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1 = \boxed{\pi/4 - \frac{1}{2}\ln 2}$$

Inverse Trig
Logarithmic
Algebraic
Trig.
Exponential

Example. Find $\int_1^2 x^4(\ln x)^2\,dx$

$u=(\ln x)^2 \quad dv=x^4\,dx$

$du=2\ln x\cdot\frac{1}{x}\,dx \quad v=\frac{x^5}{5}$

$$\int_1^2 x^4(\ln x)^2\,dx = \frac{x^5}{5}\cdot(\ln x)^2\Big]_1^2 - \frac{2}{5}\int_1^2 x^4\ln x\,dx$$

$$= \frac{32}{5}\cdot(\ln 2)^2 - \frac{1}{5}(\ln 1)^2 - \frac{2}{5}\left[\frac{x^5}{5}\cdot\ln x\Big]_1^2 - \frac{1}{5}\int_1^2 x^4\,dx\right]$$

$u=\ln x \quad dv=x^4\,dx$

$du=\frac{1}{x}\,dx \quad v=\frac{x^5}{5}$

$$= \frac{32}{5}\cdot(\ln 2)^2 - \frac{2}{5}\left[\frac{32}{5}\cdot\ln 2 - \frac{1}{5}\cdot\ln 1 - \frac{1}{5}\cdot\frac{x^5}{5}\Big]_1^2\right]$$

$$= \frac{32}{5}\cdot(\ln 2)^2 - \frac{64}{25}\cdot\ln 2 + \frac{2}{25}\left[\frac{32}{5} - \frac{1}{5}\right]$$

$$= \boxed{\frac{32}{5}\cdot(\ln 2)^2 - \frac{64}{25}\cdot\ln 2 + \frac{62}{125}}$$

Example. Find $\int_0^1 (x^2+1)e^{-x}\,dx$

$u=x^2+1 \quad dv=e^{-x}\,dx$

$du=2x\,dx \quad v=-e^{-x}$

$$\int_0^1 (x^2+1)e^{-x}\,dx = -e^{-x}\cdot(x^2+1)\Big]_0^1 + 2\int_0^1 x e^{-x}\,dx$$

$$= -e^{-1}\cdot(1+1) + e^0\cdot(0+1) + 2\left[-xe^{-x}\Big]_0^1 + \int_0^1 e^{-x}\,dx\right]$$

$u=x \quad dv=e^{-x}\,dx$

$du=dx \quad v=-e^{-x}$

$$= -2e^{-1} + 1 + 2\left[-e^{-1} + 0 - \left[e^{-x}\right]_0^1\right]$$

$$= -2e^{-1} + 1 - 2e^{-1} - 2(e^{-1} - e^0)$$

$$= \boxed{-6e^{-1} + 3}$$

Homework 7.1: Integration by Parts

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the integral.

   (a) $\int x \cos 5x \, dx$

   Let $u = x$ and $dv = \cos 5x \, dx$. Then, $du = dx$ and $v = \frac{1}{5} \sin 5x$. Then,

   $$\int x \cos 5x \, dx = \frac{x \sin 5x}{5} - \int \frac{1}{5} \sin 5x \, dx = \boxed{\frac{x \sin 5x}{5} + \frac{1}{25} \cos 5x + C}$$

   (b) $\int te^{-3t} \, dt$

   Let $u = t$ and $dv = e^{-3t} \, dt$. Then, $du = dt$ and $v = -\frac{1}{3} e^{-3t}$. Then,

   $$\int te^{-3t} \, dt = -\frac{te^{-3t}}{3} + \int \frac{1}{3} e^{-3t} \, dt = \boxed{-\frac{te^{-3t}}{3} - \frac{1}{9} e^{-3t} + C}$$

   (c) $\int (x^2 + 2x) \cos x \, dx$

   Let $u = x^2 + 2x$ and $dv = \cos x \, dx$. Then, $du = (2x + 2) \, dx$ and $v = \sin x$. Then,

   $$\int (x^2 + 2x) \cos x \, dx = (x^2 + 2x) \sin x - \int (2x + 2) \sin x \, dx.$$

   Let $U = 2x + 2$ and $dV = \sin x \, dx$. Then, $dU = 2 \, dx$ and $V = -\cos x$. Then,

   $$\int (2x + 2) \sin x \, dx = -(2x + 2) \cos x + \int 2 \cos x \, dx = -(2x + 2) \cos x + 2 \sin x + C$$

   Hence,

   $$\int (x^2 + 2x) \cos x \, dx = (x^2 + 2x) \sin x - \int (2x + 2) \sin x \, dx = (x^2 + 2x) \sin x + (2x + 2) \cos x - 2 \sin x + C = \boxed{(x^2 + 2x - 2) \sin x + (2x + 2) \cos x + C}$$

(d) $\int t^4 \ln t \, \mathrm{d}t$

Let $u = \ln t$ and $\mathrm{d}v = t^4 \, \mathrm{d}t$. Then, $\mathrm{d}u = \frac{1}{t} \, \mathrm{d}t$ and $v = \frac{t^5}{5}$. Then,

$$\int t^4 \ln t \, \mathrm{d}t = \frac{t^5 \ln t}{5} - \int \frac{t^4}{5} \, \mathrm{d}t = \boxed{\frac{t^5 \ln t}{5} - \frac{t^5}{25} + C}$$

(e) $\int (\ln x)^2 \, \mathrm{d}x$

Let $u = (\ln x)^2$ and $\mathrm{d}v = \mathrm{d}x$. Then, $\mathrm{d}u = \frac{2 \ln x}{x} \, \mathrm{d}x$ and $v = x$. Then,

$$\int (\ln x)^2 \, \mathrm{d}x = x(\ln x)^2 - \int 2 \ln x \, \mathrm{d}x$$

Let $U = \ln x$ and $\mathrm{d}V = \mathrm{d}x$. Then, $\mathrm{d}U = \frac{1}{x} \, \mathrm{d}x$ and $V = x$. Then,

$$\int (\ln x)^2 \, \mathrm{d}x = x(\ln x)^2 - 2\left[x \ln x - \int x \, \mathrm{d}x\right] = \boxed{x(\ln x)^2 - 2x \ln x + 2x + C}$$

(f) $\int e^{2\theta} \sin 3\theta \, \mathrm{d}\theta$

Let $u = \sin 3\theta$ and $\mathrm{d}v = e^{2\theta} \, \mathrm{d}\theta$. Then, $\mathrm{d}u = 3 \cos 3\theta \, \mathrm{d}\theta$ and $v = \frac{e^{2\theta}}{2}$. Then,

$$\int e^{2\theta} \sin 3\theta \, \mathrm{d}\theta = \frac{e^{2\theta} \sin 3\theta}{2} - \int \frac{3e^{2\theta} \cos 3\theta}{2}$$

Let $U = \cos 3\theta$ and $\mathrm{d}V = e^{2\theta} \, \mathrm{d}\theta$. Then, $\mathrm{d}U = -3 \sin 3\theta \, \mathrm{d}\theta$ and $V = \frac{e^{2\theta}}{2}$. Hence,

$$\begin{aligned} \int e^{2\theta} \sin 3\theta \, \mathrm{d}\theta &= \frac{e^{2\theta} \sin 3\theta}{2} - \frac{3}{2}\left[\frac{e^{2\theta} \cos 3\theta}{2} + \int \frac{3e^{2\theta} \sin 3\theta}{2} \, \mathrm{d}\theta\right] \\ &= \frac{e^{2\theta} \sin 3\theta}{2} - \frac{3e^{2\theta} \cos 3\theta}{4} - \frac{9}{4} \int e^{2\theta} \sin 3\theta \, \mathrm{d}\theta \end{aligned}$$

This implies that,

$$\frac{13}{4} \int e^{2\theta} \sin 3\theta \, \mathrm{d}\theta = \frac{e^{2\theta} \sin 3\theta}{2} - \frac{3e^{2\theta} \cos 3\theta}{4}$$

Hence,

$$\int e^{2\theta} \sin 3\theta \, \mathrm{d}\theta = \boxed{\frac{2e^{2\theta} \sin 3\theta}{13} - \frac{3e^{2\theta} \cos 3\theta}{13} + C}$$

(g) $\int z^3 e^z \, \mathrm{d}z$

Let $u = z^3$ and $\mathrm{d}v = e^z \, \mathrm{d}z$. Then, $\mathrm{d}u = 3z^2 \, \mathrm{d}z$ and $v = e^z$. Then,

$$\int z^3 e^z \, \mathrm{d}z = z^3 e^z - \int 3 \int z^2 e^z \, \mathrm{d}z$$

Let $U = z^2$ and $dV = e^z\,\mathrm{d}z$. Then, $\mathrm{d}U = 2z\,\mathrm{d}z$ and $V = e^z\,\mathrm{d}z$. Then,

$$\int z^3 e^z\,\mathrm{d}z = z^3e^z - 3\Big[z^2e^z - \int 2ze^z\,\mathrm{d}z\Big] = z^3e^z - 3z^2e^z + 6\int ze^z\,\mathrm{d}z$$

Let $u_1 = z$ and $\mathrm{d}v_1 = e^z\,\mathrm{d}z$. Then, $\mathrm{d}u_1 = \mathrm{d}z$ and $v_1 = e^z\,\mathrm{d}z$. Hence,

$$\int z^3 e^z\,\mathrm{d}z = z^3e^z - 3z^2e^z + 6\int ze^z\,\mathrm{d}z = z^3e^z - 3z^2e^z + 6(ze^z - \int e^z\,\mathrm{d}z) = \boxed{z^3e^z - 3z^2e^z + 6ze^z - 6e^z + C}$$

(h) $\int_0^{\frac{1}{2}} x\cos\pi x\,\mathrm{d}x$

Let $u = x$ and $\mathrm{d}v = \cos\pi x\,\mathrm{d}x$. Then, $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{\sin\pi x}{\pi}$. Then,

$$\int_0^{\frac{1}{2}} x\cos\pi x\,\mathrm{d}x = \Big[\tfrac{x\sin\pi x}{\pi}\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \tfrac{\sin\pi x}{\pi}\,\mathrm{d}x = \tfrac{\frac{1}{2}\sin\frac{\pi}{2}}{\pi} - \tfrac{0\sin 0}{\pi} + \Big[\tfrac{\cos\pi x}{\pi^2}\Big]_0^{\frac{1}{2}}$$
$$= \tfrac{1}{2\pi} + \tfrac{1}{\pi^2}\Big(\cos\tfrac{\pi}{2} - \cos 0\Big) = \boxed{\frac{1}{2\pi} - \frac{1}{\pi^2}}$$

(i) $\int_1^5 \frac{\ln R}{R^2}\,\mathrm{d}R$

Let $u = \ln R$ and $\mathrm{d}v = R^{-2}\,\mathrm{d}R$. Then, $\mathrm{d}u = \frac{1}{R}\,\mathrm{d}R$ and $v = -\frac{1}{R}\,\mathrm{d}R$. Then,

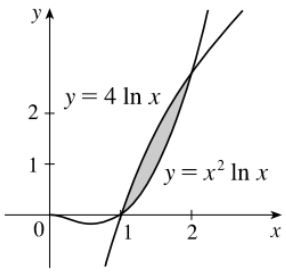
$$\int_1^5 \tfrac{\ln R}{R^2}\,\mathrm{d}R = \Big[-\tfrac{\ln R}{R}\Big]_1^5 + \int_1^5 R^{-2}\,\mathrm{d}R = -\tfrac{\ln 5}{5} + \tfrac{\ln 1}{1} + \Big[R^{-1}\Big]_1^5$$
$$= -\tfrac{\ln 5}{5} + 0 - \Big(\tfrac{1}{5} - 1\Big) = \boxed{\frac{4}{5} - \frac{\ln 5}{5}}$$

(j) $\int_0^{\frac{\pi}{3}} \sin x\ln(\cos x)\,\mathrm{d}x$

Let $u = \ln(\cos x)$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$. Then, $\mathrm{d}u = -\frac{\sin x}{\cos x}\,\mathrm{d}x$ and $v = -\cos x$. Then,

$$\int_0^{\frac{\pi}{3}} \sin x\ln(\cos x)\,\mathrm{d}x = \Big[-\cos x\ln(\cos x)\Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \sin x\,\mathrm{d}x$$
$$= -\cos\tfrac{\pi}{3}\ln\Big(\cos\tfrac{\pi}{3}\Big) + \cos 0\ln(\cos 0) + \Big[\cos x\Big]_0^{\frac{\pi}{3}}$$
$$= -\tfrac{1}{2}\ln\Big(\tfrac{1}{2}\Big) + \ln 1 + \cos\tfrac{\pi}{3} - \cos 0 = -\tfrac{1}{2}\ln 2 + \tfrac{1}{2} - 1 = \boxed{\frac{\ln 2}{2} - \frac{1}{2}}$$

2. Find the area of the region bounded by the curves $y = x^2 \ln x$, $y = 4 \ln x$.



First we find the points of intersection:

$$x^2 \ln x = 4 \ln x$$
$$x^2 \ln x - 4 \ln x = 0$$
$$\ln x(x^2 - 4) = 0$$

Hence, $\ln x = 0$ and $x^2 - 4 = 0$. Thus, $x = e^0 = 1$ and $x = \pm 2$. The shaded region is in the interval $1 \leq x \leq 2$. Then,

$$A = \int_1^2 4 \ln x - x^2 \ln x \, dx = \int_1^2 (4 - x^2) \ln x \, dx$$

Let $u = lnx$ and $dv = (4 - x^2)\, dx$. Then, $du = \frac{1}{x}\, dx$ and $v = \left(4x - \frac{x^3}{3}\right)$. Then,

$$A = \int_1^2 4 \ln x - x^2 \ln x \, dx = \int_1^2 (4 - x^2) \ln x \, dx$$
$$= \left[4x \ln x - \frac{x^3 \ln x}{3}\right]_1^2 - \int_1^2 \frac{1}{x}\left(4x - \frac{x^3}{3}\right) dx = 8 \ln 2 - \frac{8 \ln 2}{3} - \ln 1 + \frac{\ln 1}{3} - \int_1^2 4 - \frac{x^2}{3}\, dx$$
$$= 8 \ln 2 - \frac{8 \ln 2}{3} - \left[4x - \frac{x^3}{9}\right]_1^2 = 8 \ln 2 - \frac{8 \ln 2}{3} - \left(8 - \frac{8}{9} - 4 + \frac{1}{9}\right) = \boxed{\frac{16 \ln 2}{3} - \frac{29}{9}}$$

7.2 Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We will use the following identities:

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin x \cos x = \frac{1}{2}\sin 2x$

Strategy for evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Then substitute $u = \sin x$.

(b) If the power of sine is odd, save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine. Then substitute $u = \cos x$.

(c) If the powers of both sine and cosine are even, use the half-angle identities.

If the powers of both sine and cosine are odd, either (a) or (b) can be used.

Example. Find $\int \sin^4 x \cos^3 x \, dx$.

$$\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cdot \underbrace{\cos x \, dx}_{\text{"}du\text{"}} = \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$
$du = \cos x \, dx$

$$= \int u^4 (1 - u^2) \, du = \int u^4 - u^6 \, du = \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \boxed{\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C}$$

Example. Find $\int \sin^5 x\,dx$.

$$\int \sin^5 x\,dx = \int \sin^4 x \cdot \sin x\,dx = \int \sin^2 x \cdot \sin^2 x \cdot \sin x\,dx$$

"du"

$$= \int (1-\cos^2 x)(1-\cos^2 x)\sin x\,dx = -\int (1-u^2)(1-u^2)\,du$$

$u = \cos x$, $du = -\sin x\,dx$

$$= -\int 1 - 2u^2 + u^4\,du$$

$$= -\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C$$

$$= \boxed{-\left[\cos x - \frac{2\cos^3 x}{3} + \frac{\cos^5 x}{5}\right] + C}$$

Example. Evaluate $\int_0^\pi \sin^2 x\,dx$.

Note: $\int_0^\pi \sin^2 x\,dx = \int_0^\pi 1-\cos^2 x\,dx$ is not helpful!

$$\int_0^\pi \sin^2 x\,dx = \int_0^\pi \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int_0^\pi 1-\cos 2x\,dx$$

$$= \frac{1}{2}\int_0^\pi 1\,dx - \frac{1}{2}\int_0^\pi \cos 2x\,dx = \frac{1}{2}\cdot x\Big]_0^\pi - \frac{1}{4}\int_{x=0}^{x=\pi} \cos u\,du$$

$u = 2x$, $du = 2\,dx$, $\frac{1}{2}du = dx$

$$= \frac{1}{2}\cdot\pi - \frac{1}{4}\sin u\Big]_{x=0}^{x=\pi}$$

$$= \frac{\pi}{2} - \frac{1}{4}\sin(2x)\Big]_0^\pi = \pi/2 - \frac{1}{4}\sin 2\pi + \frac{1}{4}\sin 0 = \boxed{\pi/2}$$

Example. Find $\int \sin^4 x\,dx$.

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]^2\,dx$$

$$= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2(2x)\,dx = \frac{1}{4}\int 1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\,dx$$

Note: $\cos^2(2x) = \frac{1}{2}(1+\cos 4x)$

$$= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\,dx$$

$$= \frac{1}{4}\left[\frac{3}{2}\cdot x - 2\cdot\frac{\sin(2x)}{2} + \frac{1}{2}\cdot\frac{\sin(4x)}{4}\right] + C$$

$$= \boxed{\frac{3}{8}\cdot x - \frac{1}{4}\cdot\sin(2x) + \frac{1}{32}\cdot\sin(4x) + C}$$

Now we will look at integrands which are combinations of powers of tangent and secant.

Strategy for evaluating $\int \tan^m x \sec^n x \, dx$

For "du"

(a) If the power of secant is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$. Then substitute $u = \tan x$.

(b) If the power of tangent is odd, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in term of $\sec x$. Then $u = \sec x$, "du"

For other cases, the guidelines are not as clear-cut. We might need the following integrals:

- $\boxed{\int \tan x \, dx = \ln|\sec x| + C}$
- $\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$

Example. Evaluate $\int \tan^8 x \sec^6 x \, dx$.

"du"

$$\int \tan^8 x \sec^6 x \, dx = \int \tan^8 x \cdot \sec^4 x \cdot \sec^2 x \, dx = \int \tan^8 x \cdot (\sec^2 x)^2 \cdot \sec^2 x \, dx$$

$$= \int \tan^8 x \cdot (1 + \tan^2 x)^2 \cdot \sec^2 x \, dx = \int u^8 (1+u^2)^2 \, du$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^8 (1 + 2u^2 + u^4) \, du = \int u^8 + 2u^{10} + u^{12} \, du$$

$$= \frac{u^9}{9} + \frac{2u^{11}}{11} + \frac{u^{13}}{13} + C$$

$$= \boxed{\frac{\tan^9 x}{9} + \frac{2\tan^{11} x}{11} + \frac{\tan^{13} x}{13} + C}$$

Example. Find $\int \tan^5 x \sec^9 x \, dx$.

"du"

$\int \tan^5 x \sec^9 x \, dx = \int \tan^4 x \sec^8 x \cdot \sec x \tan x \, dx$

$= \int (\tan^2 x)^2 \cdot \sec^8 x \cdot \sec x \tan x \, dx$

$= \int (\sec^2 x - 1)^2 \cdot \sec^8 x \cdot \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2 - 1)^2 \cdot u^8 \, du = \int (u^4 - 2u^2 + 1) \cdot u^8 \, du$

$= \int u^{12} - 2u^{10} + u^8 \, du = \frac{u^{13}}{13} - \frac{2u^{11}}{11} + \frac{u^9}{9} + C$

$= \boxed{\frac{\sec^{13} x}{13} - \frac{2\sec^{11} x}{11} + \frac{\sec^9 x}{9} + C}$

Example. Find $\int \tan^3 x \, dx$.

* strategy doesn't apply

$\int \tan^3 x \, dx = \int \tan^2 x \cdot \tan x \, dx = \int (\sec^2 x - 1) \cdot \tan x \, dx$

$= \int \sec^2 x \cdot \tan x \, dx - \int \tan x \, dx = \int u \, du - \ln|\sec x|$

$= \frac{u^2}{2} - \ln|\sec x| + C$

$= \boxed{\frac{\tan^2 x}{2} - \ln|\sec x| + C}$

$u = \tan x$

$du = \sec^2 x \, dx$

($u = \sec x$ also works)

Example. Find $\int \sec^3 x \, dx$.

$$\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx = \sec x \tan x$$

$$u = \sec x \qquad dv = \sec^2 x\,dx \qquad - \int \sec x \tan^2 x\,dx$$

$$du = \sec x \tan x\,dx \qquad v = \tan x$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\Rightarrow 2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

Integrals of the form $\int \cot^m x \csc^n x \, dx$ can be found by similar methods by using the identity $1 + \cot^2 x = \csc^2 x$ and $(\cot x)' = -\csc^2 x$ and $(\csc x)' = -\csc x \cot x$.

Example. Evaluate $\int \cot^6 x \csc^4 x \, dx$.

$\int \cot x\,dx = \ln|\sin x| + C$

$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

"du"

$$\int \cot^6 x \csc^4 x\,dx = \int \cot^6 x \cdot \csc^2 x \cdot \csc^2 x\,dx$$

$$= \int \cot^6 x (1 + \cot^2 x) \cdot \csc^2 x\,dx = -\int u^6 (1+u^2)\,du$$

$$= -\int u^6 + u^8\,du = -\left(\frac{u^7}{7} + \frac{u^9}{9}\right) + C$$

$u = \cot x$

$du = -\csc^2 x\,dx$

$$= \boxed{-\frac{\cot^7 x}{7} - \frac{\cot^9 x}{9} + C}$$

Homework 7.2: Trigonometric Integrals

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the integral.

   (a) $\int \sin^3\theta \cos^4\theta \, d\theta$

   $$\int \sin^3\theta \cos^4\theta \, d\theta = \int \cos^4\theta \sin^2\theta \sin\theta \, d\theta = \int \cos^4\theta(1-\cos^2\theta)\sin\theta \, d\theta$$
   $$= \int (\cos^4\theta - \cos^6\theta)\sin\theta \, d\theta$$

   Let $u = \cos\theta$. Then, $du = -\sin\theta \, d\theta$ and

   $$\int (\cos^4\theta - \cos^6\theta)\sin\theta \, d\theta = \int -(u^4 - u^6)\, du = -\frac{u^5}{5} + \frac{u^7}{7} + C$$
   $$= \boxed{-\frac{\cos^5\theta}{5} + \frac{\cos^7\theta}{7} + C}$$

   (b) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$

   $$\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = \int_0^{\frac{\pi}{2}} \sin^4 x \sin x \, dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)^2 \sin x \, dx$$
   $$= \int_0^{\frac{\pi}{2}} (1 - 2\cos^2 x + \cos^4 x)\sin x \, dx$$

   Let $u = \cos x$. Then, $du = -\sin x \, dx$. If $x = 0$ then $u = \cos 0 = 1$ and if $x = \frac{\pi}{2}$ then $u = \cos\frac{\pi}{2} = 0$. Hence,

   $$\int_0^{\frac{\pi}{2}} (1 - 2\cos^2 x + \cos^4 x)\sin x \, dx = \int_1^0 -(1 - 2u^2 + u^4)\, du =$$
   $$\int_0^1 1 - 2u^2 + u^4 \, du = \left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \boxed{\frac{8}{15}}$$

   (c) $\int t\cos^5(t^2)\, dt$

   $$\int t\cos^5(t^2)\, dt = \int t\cos^4(t^2)\cos(t^2)\, dt = \int t[1-\sin^2(t^2)]^2\cos(t^2)\, dt$$

   Let $u = \sin(t^2)$. Then, $du = 2t\cos(t^2)\, dt$

$$\int t[1-\sin^2(t^2)]^2\cos(t^2)\,dt = \int \tfrac{1}{2}\Big(1-u^2)^2\,du = \tfrac{1}{2}\int(1-2u^2+u^4)\,du =$$
$$\tfrac{1}{2}\Big[u-\tfrac{2u^3}{3}+\tfrac{u^5}{5}\Big]+C = \boxed{\frac{1}{2}\Big(\sin(t^2)-\frac{2\sin^3(t^2)}{3}+\frac{\sin^5(t^2)}{5}\Big)+C}$$

(d) $\int_0^\pi \sin^2 t\cos^4 t\,dt$

$$\int_0^\pi \sin^2 t\cos^4 t\,dt = \int_0^\pi \sin^2 t\cos^2 t\cos^2 t\,dt = \int_0^\pi (\sin t\cos t)^2\cos^2 t\,dt$$
$$= \int_0^\pi \Big(\tfrac{1}{2}\sin(2t)\Big)^2\tfrac{1}{2}\Big(1+\cos 2t\Big)\,dt = \tfrac{1}{8}\int_0^\pi \sin^2(2t)+\sin^2(2t)\cos(2t)\,dt$$
$$= \tfrac{1}{8}\int_0^\pi \sin^2(2t)\,dt + \tfrac{1}{8}\int_0^\pi \sin^2(2t)\cos(2t)\,dt$$
$$= \tfrac{1}{8}\int_0^\pi \tfrac{1}{2}\Big(1-\cos(4t)\Big)\,dt + \tfrac{1}{8}\int_0^\pi \sin^2(2t)\cos(2t)\,dt$$
$$= \tfrac{1}{16}\int_0^\pi 1-\cos(4t)\,dt + \tfrac{1}{8}\int_0^\pi \sin^2(2t)\cos(2t)\,dt$$

We can find the integral of $\cos(4t)$ by a simple u-substitution ($u=4t$, $du=4\,dt$), then $\frac{1}{16}\int_0^\pi 1-\cos(4t)\,dt = \frac{1}{16}\Big[t-\frac{\sin(4t)}{4}\Big]_0^\pi = \frac{1}{16}\Big(\pi-\frac{\sin 4\pi}{4}-0+0\Big) = \frac{\pi}{16}$
For the second integral, let $u=\sin(2t)$, then $du = 2\cos(2t)\,dt$. If $t=0$, then $u=\sin 0 = 0$ and if $t=\pi$ then $u=\sin 2\pi = 0$. Note that $\frac{1}{8}\int_0^\pi \sin^2(2t)\cos(2t)\,dt = \frac{1}{16}\int_0^0 u^2\,dt = 0$ because the limits of integration are the same. Hence,

$$\int_0^\pi \sin^2 t\cos^4 t\,dt = \tfrac{1}{16}\int_0^\pi 1-\cos(4t)\,dt + \tfrac{1}{8}\int_0^\pi \sin^2(2t)\cos(2t)\,dt = \boxed{\frac{\pi}{16}}$$

(e) $\int \tan^2 x\cos^3 x\,dx$

$$\int \tan^2 x\cos^3 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\cos^3 x\,dx = \int \sin^2 x\cos x\,dx$$

Let $u=\sin x$, then $du=\cos x\,dx$ and

$$\int \sin^2 x\cos x\,dx = \int u^2\,du = \tfrac{u^3}{3}+C = \boxed{\frac{\sin^3 x}{3}+C}$$

(f) $\int \tan^2\theta\sec^4\theta\,d\theta$

$$\int \tan^2\theta\sec^4\theta\,d\theta = \int \tan^2\theta\sec^2\theta\sec^2\theta\,d\theta = \int \tan^2\theta(\tan^2\theta+1)\sec^2\theta\,d\theta$$
$$= \int(\tan^4\theta+\tan^2\theta)\sec^2\theta\,d\theta$$

Let $u=\tan\theta$, then $du=\sec^2\theta\,d\theta$. Hence,

$$\int(\tan^4\theta+\tan^2\theta)\sec^2\theta\,d\theta = \int(u^4+u^2)\,du = \tfrac{u^5}{5}+\tfrac{u^3}{3}+C = \boxed{\frac{\tan^5\theta}{5}+\frac{\tan^3\theta}{3}+C}$$

(g) $\int_0^{\frac{\pi}{4}} \tan^4 t \, \mathrm{d}t$

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^4 t \, \mathrm{d}t &= \int_0^{\frac{\pi}{4}} \tan^2 t \tan^2 t \, \mathrm{d}t = \int_0^{\frac{\pi}{4}} \tan^2 t(\sec^2 t - 1) \, \mathrm{d}t \\ &= \int_0^{\frac{\pi}{4}} \tan^2 t \sec^2 t \, \mathrm{d}t - \int_0^{\frac{\pi}{4}} \tan^2 t \, \mathrm{d}t \\ &= \int_0^{\frac{\pi}{4}} \tan^2 t \sec^2 t \, \mathrm{d}t - \int_0^{\frac{\pi}{4}} \sec^2 t - 1 \, \mathrm{d}t\end{aligned}$$

To find $\int_0^{\frac{\pi}{4}} \tan^2 t \sec^2 t \, \mathrm{d}t$, let $u = \tan t$, then $\mathrm{d}u = \sec^2 t \, \mathrm{d}t$. If $t = 0$, then $u = \tan 0 = 0$ and if $t = \frac{\pi}{4}$, then $u = \tan \frac{\pi}{4} = 1$. Hence,

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^2 t \sec^2 t \, \mathrm{d}t - \int_0^{\frac{\pi}{4}} \sec^2 t - 1 \, \mathrm{d}t &= \int_0^1 u^2 \, \mathrm{d}u - \Big[\tan t - t\Big]_0^{\frac{\pi}{4}} \\ = \Big[\frac{u^3}{3}\Big]_0^1 - \Big(\tan \tfrac{\pi}{4} - \tfrac{\pi}{4} - 0 + 0\Big) &= \tfrac{1}{3} - 1 + \tfrac{\pi}{4} = \boxed{\frac{\pi}{4} - \frac{2}{3}}\end{aligned}$$

(h) $\int \frac{\sin x}{\cos^3 x} \, \mathrm{d}x$

Let $u = \cos x$, then $\mathrm{d}u = -\sin x \, \mathrm{d}x$. Hence,

$$\int \frac{\sin x}{\cos^3 x} \, \mathrm{d}x = -\int \frac{1}{u^3} \, \mathrm{d}u = -\int u^{-3} \, \mathrm{d}u = -\frac{u^{-2}}{-2} + C = \frac{1}{2\cos^2 x} + C = \boxed{\frac{\sec^2 x}{2} + C}$$

Alternatively,

$$\int \frac{\sin x}{\cos^3 x} \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} \, \mathrm{d}x = \int \tan x \sec^2 x \, \mathrm{d}x$$

Let $u = \tan x$, then $\mathrm{d}u = \sec^2 x \, \mathrm{d}x$. Hence,

$$\int \tan x \sec^2 x \, \mathrm{d}x = \int u \, \mathrm{d}u = \frac{u^2}{2} + C = \boxed{\frac{\tan^2 x}{2} + C}$$

(i) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x \, \mathrm{d}x$

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x \, \mathrm{d}x &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \cdot \cot^2 x \, \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x(\csc^2 x - 1) \, \mathrm{d}x \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \csc^2 x \, \mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, \mathrm{d}x \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \csc^2 x \, \mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} \, \mathrm{d}x\end{aligned}$$

To find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \csc^2 x \, \mathrm{d}x$, let $u = \cot x$, then $\mathrm{d}u = -\csc^2 x \, \mathrm{d}x$. If $x = \frac{\pi}{4}$, then $u = \cot \frac{\pi}{4} = 1$ and if $x = \frac{\pi}{2}$, then $u = \cot \frac{\pi}{2} = 0$.
To find $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} \, \mathrm{d}x$, let $U = \sin x$, then $\mathrm{d}U = \cos x \, \mathrm{d}x$. If If $x = \frac{\pi}{4}$, then $U = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and if $x = \frac{\pi}{2}$, then $U = \sin \frac{\pi}{2} = 1$ Hence,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \csc^2 x \, \mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} \, \mathrm{d}x = \int_1^0 -u \, \mathrm{d}u - \int_{\frac{\sqrt{2}}{2}}^1 \frac{1}{U} \, \mathrm{d}U = \left[\frac{u^2}{2}\right]_0^1 - \left[\ln|U|\right]_{\frac{\sqrt{2}}{2}}^1 =$$

$$\frac{1}{2} - \ln 1 + \ln \frac{\sqrt{2}}{2} = \boxed{\frac{1}{2} + \ln \frac{\sqrt{2}}{2}}$$

(j) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^4 x \cot^4 x \, \mathrm{d}x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^4 x \cot^4 x \, \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x \csc^2 x \csc^2 x \, \mathrm{d}x$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x (\cot^2 x + 1) \csc^2 x \, \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^6 x + \cot^4 x) \csc^2 x \, \mathrm{d}x$$

Let $u = \cot x$, then $\mathrm{d}u = -\csc^2 x \, \mathrm{d}x$. If $x = \frac{\pi}{4}$, then $u = \cot \frac{\pi}{4} = 1$ and if $x = \frac{\pi}{2}$, then $u = \cot \frac{\pi}{2} = 0$. Hence,

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^6 x + \cot^4 x) \csc^2 x \, \mathrm{d}x = -\int_1^0 (u^6 + u^4) \, \mathrm{d}u = \int_0^1 u^6 + u^4 \, \mathrm{d}u =$$

$$\left[\frac{u^7}{7} + \frac{u^5}{5}\right]_0^1 = \frac{1}{7} + \frac{1}{5} = \boxed{\frac{12}{35}}$$

7.3 Trigonometric Substitution

We will use the following trigonometric substitutions:

- Given $\sqrt{a^2 - x^2}$, let $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then use the identity $1 - \sin^2\theta = \cos^2\theta$.
- Given $\sqrt{a^2 + x^2}$, let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then use the identity $1 + \tan^2\theta = \sec^2\theta$.
- Given $\sqrt{x^2 - a^2}$, let $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then use the identity $\sec^2\theta - 1 = \tan^2\theta$.

In case we end up with inverse trig. functions

Example. Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$.

$\sqrt{9-x^2} \neq \sqrt{9} - \sqrt{x^2}$ $\quad \sqrt{9-x^2} = \sqrt{3^2 - x^2} \Rightarrow x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2} \cdot 3\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\,d\theta = \frac{1}{3}\int \frac{\sqrt{9(1-\sin^2\theta)}\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{3}\int \frac{\sqrt{9\cos^2\theta}\cdot\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{3}\int \frac{3\cos\theta\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int (\csc^2\theta - 1)\,d\theta$$

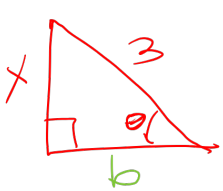
$= -\cot\theta - \theta + C$

$$= \boxed{-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

~~$u = \cot\theta$~~ $\quad du = -\csc^2\theta\,d\theta$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\quad \sin^{-1}\left(\frac{x}{3}\right) = \sin^{-1}(\sin\theta) = \theta$



$x^2 + b^2 = 3^2$

$b^2 = 9 - x^2$

$b = \sqrt{9-x^2}$

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Example. Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

$\sqrt{x^2+4} = \sqrt{x^2+2^2} \Rightarrow x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}} \cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\,d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{4\sec^2\theta}}\,d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\cdot 2\sec\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

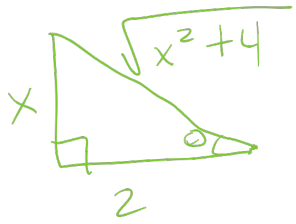
or $u = \sin\theta$, $du = \cos\theta\,d\theta$

$$= \frac{1}{4}\int \cot\theta\cdot\csc\theta\,d\theta = -\frac{1}{4}\csc\theta + C$$

$$= \boxed{-\frac{1}{4}\left(\frac{\sqrt{x^2+4}}{x}\right) + C}$$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$



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Example. Find $\int \frac{1}{\sqrt{x^2-4}}\,dx$.

$x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{\sqrt{(2\sec\theta)^2-4}} \cdot 2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta-4}}\,d\theta = \int \frac{2\sec\theta\tan\theta}{\sqrt{4(\sec^2\theta-1)}}\,d\theta$$

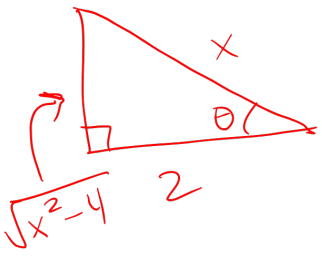
$$= \int \frac{2\sec\theta\tan\theta}{\sqrt{4\tan^2\theta}}\,d\theta = \int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta$$

$$= \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right| + C}$$

$x = 2\sec\theta$

$\frac{x}{2} = \sec\theta$



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Example. Find $\int_0^4 x^2\sqrt{16-x^2}\,dx$.

$x=4\sin\theta \Rightarrow dx=4\cos\theta\,d\theta$

$x=0 \Rightarrow 0=4\sin\theta \Rightarrow \theta=0$

$x=4 \Rightarrow 4=4\sin\theta \Rightarrow 1=\sin\theta \Rightarrow \theta=\pi/2$

$$\int_0^4 x^2\sqrt{16-x^2}\,dx = \int_0^{\pi/2} (4\sin\theta)^2\sqrt{16-(4\sin\theta)^2}\cdot 4\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 16\sin^2\theta\sqrt{16-16\sin^2\theta}\cdot 4\cos\theta\,d\theta$$

$$= 64\int_0^{\pi/2} \sin^2\theta\sqrt{16(1-\sin^2\theta)}\cdot\cos\theta\,d\theta$$

$$= 64\int_0^{\pi/2} \sin^2\theta\sqrt{16\cos^2\theta}\cdot\cos\theta\,d\theta$$

$$= 64\int_0^{\pi/2} \sin^2\theta\cdot 4\cos\theta\cdot\cos\theta\,d\theta = 256\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$= 256\int_0^{\pi/2}\left(\frac{1}{2}\sin 2\theta\right)^2 d\theta$$

$$= 256\int_0^{\pi/2}\frac{1}{4}\sin^2 2\theta\,d\theta$$

$$= 64\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 32\int_0^{\pi/2} 1-\cos 4\theta\,d\theta$$

$$= 32\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\pi/2}$$

$$= 32\left[\pi/2 - \frac{\sin 2\pi}{4} - (0-0)\right] = 32\cdot\frac{\pi}{2} = \boxed{16\pi}$$

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$\int_a^b f(x)dx = -\int_b^a f(x)dx$

Example. Find $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx$.

$(4x^2+9)^{3/2} = \left(\sqrt{4x^2+9}\right)^3 = \left(\sqrt{(2x)^2+3^2}\right)^3$

$2x = 3\tan\theta \Rightarrow x = \frac{3}{2}\tan\theta \Rightarrow dx = \frac{3}{2}\sec^2\theta\,d\theta$

If $x=0 \Rightarrow 0 = \frac{3}{2}\tan\theta \Rightarrow \theta = 0$

$x = \frac{3\sqrt{3}}{2} \Rightarrow \frac{3}{2}\sqrt{3} = \frac{3}{2}\tan\theta \Rightarrow \sqrt{3} = \tan\theta \Rightarrow \theta = \pi/3$

$$\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}dx = \int_0^{\pi/3} \frac{\left(\frac{3}{2}\tan\theta\right)^3 \cdot \frac{3}{2}\sec^2\theta\,d\theta}{\left(\sqrt{(3\tan\theta)^2+9}\right)^3} = \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta \cdot \frac{3}{2}\sec^2\theta}{\left(\sqrt{9\tan^2\theta+9}\right)^3}\,d\theta$$

$$= \frac{81}{16}\int_0^{\pi/3} \frac{\tan^3\theta\cdot\sec^2\theta}{\left(\sqrt{9(\tan^2\theta+1)}\right)^3}\,d\theta = \frac{81}{16}\int_0^{\pi/3} \frac{\tan^3\theta\cdot\sec^2\theta}{\left(\sqrt{9\sec^2\theta}\right)^3}\,d\theta$$

$$= \frac{81}{16}\int_0^{\pi/3} \frac{\tan^3\theta\sec^2\theta}{(3\sec\theta)^3}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\frac{\sin^3\theta}{\cos^3\theta}}{\frac{1}{\cos\theta}}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^2\theta\cdot\sin\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{(1-\cos^2\theta)\sin\theta}{\cos^2\theta}\,d\theta = -\frac{3}{16}\int_1^{1/2} \frac{(1-u^2)}{u^2}\,du = \frac{3}{16}\int_{1/2}^1 u^{-2}-1\,du$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$\theta = 0,\ u = \cos 0 = 1$

$\theta = \pi/3,\ u = \cos\pi/3 = 1/2$

$$= \frac{3}{16}\left[-\frac{1}{u} - u\right]_{1/2}^1 = \frac{3}{16}\left[-1-1+2+1/2\right]$$

$$= \frac{3}{16}\cdot\frac{1}{2} = \boxed{\frac{3}{32}}$$

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Homework 7.3 Solutions: Trigonometric Substitution

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

   (a) $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx,\ x = 2\sin\theta$

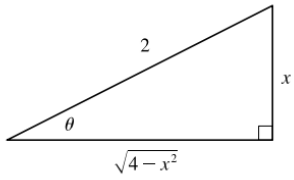
   If $x = 2\sin\theta$ then $dx = 2\cos\theta\,d\theta$. Then,

   $$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{2\cos\theta}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}\,d\theta = \int \frac{2\cos\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}\,d\theta$$

   $$= \frac{1}{2}\int \frac{\cos\theta}{\sin^2\theta\sqrt{4(1-\sin^2\theta)}}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta\sqrt{\cos^2\theta}}\,d\theta = \frac{1}{4}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \csc^2\theta\,d\theta$$

   $$= \frac{1}{4}(-\cot\theta) + C = \boxed{-\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right) + C}$$

   Note that to get the last step we used the fact that $\sin\theta = \frac{x}{2}$ and we created the right triangle with opposite side to θ equal to x, adjacent side equal to $\sqrt{4-x^2}$ and hypotenuse 2. Then, since cotangent of theta is equal to adjacent over opposite, we have $\cot\theta = \frac{\sqrt{4-x^2}}{x}$.

   (b) $\int \frac{x^3}{\sqrt{x^2+4}}\,dx,\ x = 2\tan\theta$

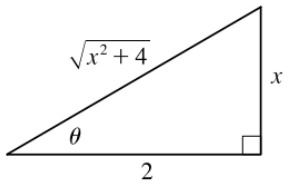
   If $x = 2\tan\theta$ then $dx = 2\sec^2\theta\,d\theta$. Then,

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{(2\tan\theta)^3 \cdot 2\sec^2\theta}{\sqrt{(2\tan\theta)^2+4}}\,d\theta = \int \frac{16\tan^3\theta\sec^2\theta}{\sqrt{4(\tan^2\theta+1)}}\,d\theta$$
$$= 8\int(\sec^2\theta - 1)\cdot\sec\theta\tan\theta\,d\theta$$

Let $u = \sec\theta$, then $du = \sec\theta\tan\theta\,d\theta$ and

$$8\int(\sec^2\theta - 1)\cdot\sec\theta\tan\theta\,d\theta = 8\int(u^2-1)\,du$$
$$= 8\left(\frac{u^3}{3} - u\right) + C = 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$
$$= 8\left[\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \frac{\sqrt{x^2+4}}{2}\right] + C = \boxed{\frac{(x^2+4)^{\frac{3}{2}}}{3} - 4\sqrt{x^2+4} + C}$$

Note that $\tan\theta = \frac{x}{2}$. Hence, we create the right triangle shown below to find $\sec\theta$:



2. Evaluate the integral

(a) $\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx$

Note that we could apply trigonometric substitution to this integral but, it would be easier to use u-substitution. Let $u = 36 - x^2$, then $du = -2x\,dx$. If $x = 0$, then $u = 36 - 0^2 = 36$ and if $x = 3$ then $u = 36 - 9 = 27$.Then,

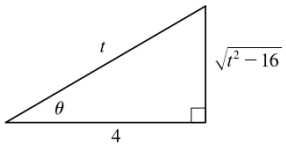
$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = -\frac{1}{2}\int_{36}^{27} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{27}^{36} u^{-\frac{1}{2}}\,du = \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{27}^{36}$$
$$= \sqrt{36} - \sqrt{27} = 6 - 3\sqrt{3}$$

(b) $\int \frac{1}{t^2\sqrt{t^2-16}}\,dt$

Let $t = 4\sec\theta$, then $dt = 4\sec\theta\tan\theta\,d\theta$ and

$$\int \frac{1}{t^2\sqrt{t^2-16}}\,dt = \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\,d\theta = \frac{1}{4}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{16\tan^2\theta}}\,d\theta$$
$$= \frac{1}{16}\int \frac{\tan\theta}{\sec\theta\tan\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta$$

$$= \tfrac{1}{16}\sin\theta + C = \boxed{\frac{\sqrt{t^2-16}}{16t} + C}$$



(c) $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,\mathrm{d}x$

Let $3x = 2\sin\theta$, then $3\,\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$, which implies $\mathrm{d}x = \frac{2}{3}\cos\theta\,\mathrm{d}\theta$. If $x = 0$, then $\theta = 0$ and if $x = \frac{2}{3}$, then $\theta = \frac{\pi}{2}$. Hence,

$$\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} 2\cos\theta \cdot \tfrac{2}{3}\cos\theta\,\mathrm{d}\theta = \tfrac{4}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta =$$
$$\tfrac{4}{3}\int_0^{\frac{\pi}{2}} \tfrac{1}{2}\Big(1+\cos 2\theta\Big)\,\mathrm{d}\theta = \tfrac{2}{3}\Big[\theta + \tfrac{\sin 2\theta}{2}\Big]_0^{\frac{\pi}{2}} = \tfrac{2}{3}\Big[\tfrac{\pi}{2} + 0 - (0+0)\Big] = \boxed{\frac{\pi}{3}}$$

(d) $\int_0^2 \frac{1}{\sqrt{4+t^2}}\,\mathrm{d}t$

Let $t = 2\tan\theta$, then $\mathrm{d}t = 2\sec^2\theta\,\mathrm{d}\theta$. If $t = 0$ then $\theta = 0$ and if $t = 2$ then $\tan\theta = 1$, which implies $\theta = \frac{\pi}{4}$. Then,

$$\int_0^2 \frac{1}{\sqrt{4+t^2}}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{2\sec^2\theta}{\sqrt{4(1+\tan^2\theta)}}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{4}} \sec\theta\,\mathrm{d}\theta = \Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\frac{\pi}{4}} = \Big|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\Big| - \ln|\sec 0 - \tan 0|$$
$$= \ln\Big|\tfrac{2}{\sqrt{2}} + 1\Big| - \ln|1-0| = \boxed{\ln(\sqrt{2}+1)}$$

(e) $\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{1}{x^5\sqrt{9x^2-1}}\,\mathrm{d}x$

Let $3x = \sec\theta$, then $3\,\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$, which implies $\mathrm{d}x = \frac{1}{3}\sec\theta\tan\theta\,\mathrm{d}\theta$. If $x = \frac{\sqrt{2}}{3}$ then $\sqrt{2} = \sec\theta$, which implies $\theta = \frac{\pi}{4}$. If $x = \frac{2}{3}$ then $2 = \sec\theta$, which implies $\theta = \frac{\pi}{3}$. Then,

$$\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{1}{x^5\sqrt{9x^2-1}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{\sec\theta\tan\theta}{3}}{(\frac{\sec\theta}{3})^5\tan\theta}\,\mathrm{d}\theta = 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\sec^5\theta}\,\mathrm{d}\theta$$

$$= 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^4\theta}\,\mathrm{d}\theta = 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4\theta\,\mathrm{d}\theta = 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2 \mathrm{d}\theta =$$
$$\tfrac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+2\cos 2\theta+\cos^2 2\theta)\,\mathrm{d}\theta$$

$$= \tfrac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]\mathrm{d}\theta = \tfrac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\,\mathrm{d}\theta$$

$$= \left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \tfrac{81}{4}\left[\tfrac{3}{2}\cdot\tfrac{\pi}{3}+\sin\tfrac{2\pi}{3}+\tfrac{1}{8}\sin\tfrac{4\pi}{3}-\left(\tfrac{3}{2}\cdot\tfrac{\pi}{4}+\sin\tfrac{\pi}{2}+\tfrac{1}{8}\sin\pi\right)\right]$$

$$= \tfrac{81}{4}\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}+\tfrac{1}{8}\cdot\left(-\tfrac{\sqrt{3}}{2}\right)-\tfrac{3\pi}{8}-1-0\right) = \boxed{\frac{81}{4}\left(\frac{\pi}{8}+\frac{7\sqrt{3}}{16}-1\right)}$$

(f) $\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$

Note that we could apply trigonometric substitution to this integral but, it would be easier to use u-substitution. Let $u = 1+x^2$ then $\mathrm{d}u = 2x\,\mathrm{d}x$ and $x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. Then,

$$\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = \tfrac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u = \tfrac{1}{2}\left(2u^{\frac{1}{2}}\right)+C = \sqrt{u}+C = \boxed{\sqrt{1+x^2}+C}$$

7.4 Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simple fractions, called **partial fractions**, that we already know how to integrate.

Method of Partial Fractions

- We start with $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with degree $P(x) <$ degree $Q(x)$.
- We factor $Q(x)$ completely and add a term for each factor.

Factor in denominator	Terms in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}$, $k = 1, 2, 3, \ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$, $k = 1, 2, 3, \ldots$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ is an integral that shows up frequently.

Example. Evaluate $\int \frac{3x+11}{x^2-x-6}\,dx$.

$\deg(3x+11) = 1 < \deg(x^2-x-6) = 2$ ✓

$\frac{3x+11}{x^2-x-6} = \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$

Goal: Find A and B

distinct linear factors

$(x-3)(x+2)\left[\frac{3x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}\right] \Rightarrow 3x+11 = A(x+2) + B(x-3)$

Plug in zeros: $x=-2$: $3(-2)+11 = A(0) + B(-5) \Rightarrow 5 = -5B \Rightarrow B = -1$

$x=3$: $3(3)+11 = A(5)+B(0) \Rightarrow 20 = 5A \Rightarrow A = 4$

7.4 Integration of Rational Functions by Partial Fractions

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simple fractions, called **partial fractions**, that we already know how to integrate.

Method of Partial Fractions

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Factor in denominator	Terms in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\ldots+\dfrac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

$\boxed{\displaystyle\int \frac{1}{x^2+a^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$ is an integral that shows up frequently.

Example. Evaluate $\displaystyle\int \frac{3x+11}{x^2-x-6}\,\mathrm{d}x$.

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{4}{x-3}-\frac{1}{x+2}\,dx = 4\int \frac{1}{x-3}\,dx - \int \frac{1}{x+2}\,dx$$

$u = x-3$, $du = dx$ $\qquad$ $U = x+2$, $dU = dx$

$$= 4\int \frac{1}{u}\,du - \int \frac{1}{U}\,dU = 4\ln|u| - \ln|U| + C$$

$$= \boxed{4\ln|x-3| - \ln|x+2| + C}$$

Example. Find $\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$.

$\deg(x^2+4)=2 \qquad \deg(3x^3+4x^2-4x)=3$

$$\frac{x^2+4}{3x^3+4x^2-4x} = \frac{x^2+4}{x(3x^2+4x-4)} = \frac{x^2+4}{x(3x-2)(x+2)} = \frac{A}{x}+\frac{B}{3x-2}+\frac{C}{x+2}$$

→ 3 distinct linear factors

$$(x)(3x-2)(x+2)\left[\frac{x^2+4}{3x^3+4x^2-4x} = \frac{A}{x}+\frac{B}{3x-2}+\frac{C}{x+2}\right]$$

$$x^2+4 = A(3x-2)(x+2)+Bx(x+2)+Cx(3x-2)$$

$x=0$: $0^2+4 = A(0-2)(0+2)+0+0 \Rightarrow 4=-4A \Rightarrow \boxed{A=-1}$

$x=-2$ $(-2)^2+4 = 0+0+C(-2)[3(-2)-2] \Rightarrow 8=16C \Rightarrow \boxed{C=1/2}$

$x=\frac{2}{3}$: $\left(\frac{2}{3}\right)^2+4 = 0+B\cdot\frac{2}{3}\left(\frac{2}{3}+2\right)+0 \Rightarrow \frac{4}{9}+4 = B\cdot\frac{2}{3}\left(\frac{8}{3}\right)$

$\Rightarrow \frac{40}{9} = \frac{16}{9}\cdot B \Rightarrow \boxed{B=\frac{40}{16}=\frac{5}{2}}$

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx = \int \frac{-1}{x}+\frac{\frac{5}{2}}{3x-2}+\frac{1/2}{x+2}\,dx$$

$$= -\int \frac{1}{x}dx + \frac{5}{2}\int \frac{1}{3x-2}\,dx + \frac{1}{2}\int \frac{1}{x+2}\,dx = -\ln|x| + \frac{1}{2}\ln|x+2|$$

$u=3x-2$, $du=3dx$, $\frac{1}{3}du=dx$

$$+\frac{5}{2}\cdot\frac{1}{3}\int\frac{1}{u}\,du = -\ln|x|+\frac{1}{2}\ln|x+2| + \frac{5}{6}\cdot\ln|u| + C$$

$$= \boxed{-\ln|x|+\frac{1}{2}\ln|x+2|+\frac{5}{6}\cdot\ln|3x-2|+C}$$

Example. Find $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$.

deg num > deg den $\Rightarrow$ long division

$\frac{x^4}{x^3} = x$

$\frac{x^3}{x^3} = 1$

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1 \overline{)\, x^4 - 2x^2 + 4x + 1} \\ -(x^4 - x^3 - x^2 + x) \\ \hline x^3 - x^2 + 3x + 1 \\ -(x^3 - x^2 - x + 1) \\ \hline 4x \end{array}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1 + \frac{4x}{x^3-x^2-x+1}$$

$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x-1)(x^2-1) = (x-1)^2(x+1)$ ↓ repeated linear factor

$$\left[\frac{4x}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right] (x-1)^2(x+1)$$

$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$

$x=1$: $4 = A\cdot 0 + B\cdot 0 + C\cdot 2 \Rightarrow C=2$

$x=-1$: $-4 = A(-2)^2 + B\cdot 0 + C\cdot 0 \Rightarrow A=-1$

$x=0$: $0 = -1\cdot(-1)^2 + B(-1)(1) + 2\cdot(1) \Rightarrow B=1$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int x+1 - \frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$u = x-1$, $du = dx$

$$= \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| + 2\int \frac{1}{u^2}\,du$$

$$= \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{u} + C$$

$$= \boxed{\frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C}$$

Example. Find $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

deg num < deg den ✓

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \quad \times x(x^2+4)$$

irreducible quadratic

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)\cdot x$$
$$= Ax^2 + 4A + Bx^2 + Cx$$
$$= Ax^2 + Bx^2 + Cx + 4A$$

$4 = 4A \Rightarrow A = 1$

$C = -1$

$A + B = 2 \Rightarrow B = 2 - A = 2 - 1 = 1$

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \ln|x| + \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$$
$$= \ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$
$$= \boxed{\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Example. Find $\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx$.

deg num = 3
deg den = 5 ✓

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+4)} + \frac{Dx+E}{(x^2+4)^2}$$ $\cdot (x-1)(x^2+4)^2$

$$x^3+10x^2+3x+36 = A(x^2+4)^2 + (Bx+C)(x-1)(x^2+4) + (Dx+E)(x-1)$$
$$= A(x^4+8x^2+16) + (Bx+C)(x^3+4x-x^2-4) + Dx^2-Dx+Ex-E$$
$$= Ax^4+8Ax^2+16A+Bx^4+4Bx^2-Bx^3-4Bx+Cx^3+4Cx-Cx^2-4C$$
$$+Dx^2-Dx+Ex-E$$
$$= (A+B)x^4+(C-B)x^3+(8A+4B-C+D)x^2+(-4B+4C-D+E)x$$
$$+16A-4C-E$$

$A+B=0 \Rightarrow A=-B$
$C-B=1 \Rightarrow C=1+B$

$8A+4B-C+D=10 \rightarrow 8(-B)+4B-(1+B)+D=10$
$-4B-1-B+D=10 \Rightarrow -5B+D=11$ ←

$-4B+4C-D+E=3 \rightarrow -4B+4(1+B)-D+E=3 \Rightarrow -D+E=-1$ ←

$16A-4C-E=36 \rightarrow 16(-B)-4(1+B)-E=36$
$-20B-E=40$ ←

$-4(-5B+D=11) \Rightarrow 20B-4D=-44$
$+\ -20B-E=40$
$-4D-E=-4$

$+\ -D+E=-1$
$-5D=-5$

$D=1$

$E=-1+D=-1+1=0$

$-5B+1=11 \Rightarrow -5B=10$
$B=-2$

$A=2$
$C=1-2=-1$

$$\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx = \int \frac{2}{x-1} + \frac{-2x-1}{x^2+4} + \frac{x}{(x^2+4)^2}\,dx$$
$$= 2\int \frac{1}{x-1}\,dx - \int \frac{2x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx + \int \frac{x}{(x^2+4)^2}\,dx$$

$u=x^2+4$
$du=2x\,dx$

$$= 2\ln|x-1| - \int \frac{1}{u}\,du - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\int \frac{1}{u^2}\,du$$

$\int \frac{1}{u}\,du = -\ln|u|$

$-\frac{1}{2}\cdot\frac{1}{u}$

$$= \boxed{2\ln|x-1| - \ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\cdot\frac{1}{x^2+4} + C}$$

Homework 7.4 Solutions: Integration of rational functions by partial fractions

Prof. Leyda Almodóvar

Name: _____

1. Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients.

   (a) $\frac{x-6}{x^2+x-6} = \frac{x-6}{x^2+x-6} = \frac{x-6}{(x+3)(x-2)} = \boxed{\frac{A}{x+3} + \frac{B}{x-2}}$

   (b) $\frac{x^2}{x^2+x+6}$

   Since the polynomial in the numerator has the same degree as the polynomial in the denominator, we perform long division first.

   We find that $\frac{x^2}{x^2+x+6} = 1 - \frac{x+6}{x^2+x+6}$. Then, since $x^2 + x + 6$ is irreducible (it cannot be factored into smaller polynomials) we get that

   $$\frac{x^2}{x^2+x+6} = \boxed{1 - \frac{Bx+C}{x^2+x+6}}$$

   (c) $\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}$

   Since the polynomial in the numerator has a higher degree than the polynomial in the denominator, we must perform long division first. We find that $\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{x^2-2x+1}$. Note that $x^2 - 2x + 1 = (x-1)^2$. Hence,

   $$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 - \frac{2x-1}{(x-1)^2} = x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

(d) $\frac{x^2-1}{x^3+x^2+x}$

Note that $\frac{x^2-1}{x^3+x^2+x} = \frac{x^2-1}{x(x^2+x+1)}$. Since x^2+x+1 is an irreducible polynomial, we get

$$\frac{x^2-1}{x^3+x^2+x} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

2. Evaluate the integral.

(a) $\int \frac{3t-2}{t+1}\,\mathrm{d}t$

Since the polynomial in the numerator has the same degree as the polynomial in the denominator, we must perform long division first.

We find that $\frac{3t-2}{t+1} = 3 - \frac{5}{t+1}$. Then,

$$\int \frac{3t-2}{t+1}\,\mathrm{d}t = \int \left(3 - \frac{5}{t+1}\right)\mathrm{d}t = \int 3\,\mathrm{d}t - 5\int \frac{1}{t+1}\,\mathrm{d}t = 3t - 5\int \frac{1}{u}\,\mathrm{d}u$$
$$= 3t - 5\ln|u| + C = \boxed{3t - 5\ln|t+1| + K}$$

Note that a simple u-substitution ($u = t+1$) was used to find the integral.

(b) $\int \frac{y}{(y+4)(2y-1)}\,\mathrm{d}y$

$\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1}$ implies $y = A(2y-1) + B(y+4)$.

If $y = -4$, then $-4 = A(-9) + B(0)$, that is, $A = \frac{4}{9}$.
If $y = \frac{1}{2}$, then $\frac{1}{2} = \frac{4}{9}\cdot(0) + B\left(\frac{9}{2}\right)$. That is, $B = \frac{1}{9}$.

$$\int \frac{y}{(y+4)(2y-1)}\,\mathrm{d}y = \int \frac{\frac{4}{9}}{y+4}\,\mathrm{d}y + \int \frac{\frac{1}{9}}{2y-1}\,\mathrm{d}y = \frac{4}{9}\int \frac{1}{y+4}\,\mathrm{d}y + \frac{1}{9}\int \frac{1}{2y-1}\,\mathrm{d}y$$
$$= \boxed{\frac{4}{9}\ln|y+4| + \frac{1}{18}\ln|2y-1| + K}$$

Note that two simple substitutions were used ($u = y+4$ and $U = 2y-1$, respectively) to find both integrals.

(c) $\int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x$

$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$ implies $x - 4 = A(x-2) + B(x-3)$.

If $x = 3$, then $-1 = A(1) + B(0)$, that is, $A = -1$.

If $x = 2$, then $-2 = -1(0) + B(-1)$, that is, $B = 2$.

$$\int_0^1 \tfrac{x-4}{x^2-5x+6}\,\mathrm{d}x = \int_0^1 \left(-\tfrac{1}{x-3} + \tfrac{2}{x-2}\right)\mathrm{d}x = \Big[-\ln|x-3| + 2\ln|x-2|\Big]_0^1$$

$$= -\ln|-2| + 2\ln|-1| - (-\ln|-3| + 2\ln|-2|) = \boxed{-3\ln 2 + \ln 3}$$

Note that two simple substitutions were used ($u = x - 3$ and $U = x - 2$, respectively) to find both integrals.

(d) $\int_1^2 \tfrac{x^3+4x^2+x-1}{x^3+x^2}\,\mathrm{d}x$

Since the polynomial in the numerator has the same degree as the polynomial in the denominator, we must perform long division first.

We find that $\tfrac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \tfrac{3x^2+x-1}{x^3+x^2}$.
Next we use the method of partial fractions on $\tfrac{3x^2+x-1}{x^3+x^2}$:

$\tfrac{3x^2+x-1}{x^3+x^2} = \tfrac{3x^2+x-1}{x^2(x+1)} = \tfrac{A}{x} + \tfrac{B}{x^2} + \tfrac{C}{x+1}$

$3x^2 + x - 1 = Ax(x+1) + B(x+1) + Cx^2$
If $x = 0$, then $-1 = A(0)(1) + B(1) + C(0)^2$, that is, $B = -1$.
If $x = -1$, then $3 - 1 - 1 = A(-1)(0) - 1(0) + C(-1)^2$, that is $C = 1$.
To find A, we substitute any value for x, for example, let $x = 1$, then
$3 = A(1)(2) - 1(2) + 1(1)^2$, that is $A = 2$.

$$\int_1^2 \tfrac{x^3+4x^2+x-1}{x^3+x^2}\,\mathrm{d}x = \int_1^2 1 + \tfrac{2}{x} - \tfrac{1}{x^2} + \tfrac{1}{x+1}\,\mathrm{d}x$$

$$= \int_1^2 1\,\mathrm{d}x + 2\int_1^2 \tfrac{1}{x}\,\mathrm{d}x - \int_1^2 x^{-2}\,\mathrm{d}x + \int_1^2 \tfrac{1}{x+1}\,\mathrm{d}x = \Big[x + 2\ln|x| + \tfrac{1}{x} + \ln|x+1|\Big]_1^2$$

$$= 2 + 2\ln 2 + \tfrac{1}{2} + \ln 3 - (1 + 2\ln 1 + 1 + \ln 2) = \boxed{\frac{1}{2} + \ln 2 + \ln 3} = \boxed{\frac{1}{2} + \ln 6}$$

(e) $\int \tfrac{x^4+9x^2+x+2}{x^2+9}\,\mathrm{d}x$

Since the polynomial in the numerator has a higher degree than the polynomial in the denominator, we must perform long division first.
We find that $\tfrac{x^4+9x^2+x+2}{x^2+9} = x^2 + \tfrac{x+2}{x^2+9} = x^2 + \tfrac{x}{x^2+9} + \tfrac{2}{x^2+9}$.

Note that the method of partial fractions wasn't necessary because $\frac{x+2}{x^2+9} = \frac{Ax+B}{x^2+9}$ so we already know $A = 1$ and $B = 2$.

$$\int \frac{x^4+9x^2+x+2}{x^2+9}\,\mathrm{d}x = \int x^2 + \frac{x}{x^2+9} + \frac{2}{x^2+9}\,\mathrm{d}x = \int x^2\,\mathrm{d}x + \int \frac{x}{x^2+9}\,\mathrm{d}x + 2\int \frac{1}{x^2+9}\,\mathrm{d}x$$
$$= \frac{x^3}{3} + \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u + 2\tan^{-1}\left(\frac{x}{3}\right) + C = \frac{x^3}{3} + \frac{1}{2}\ln|u| + 2\tan^{-1}\left(\frac{x}{3}\right) + C$$
$$= \boxed{\frac{x^3}{3} + \frac{1}{2}\ln(x^2+9) + 2\tan^{-1}\left(\frac{x}{3}\right) + K}$$

Note that u-substitution was used on the second integral ($u = x^2 + 9$, $\mathrm{d}u = 2x\,\mathrm{d}x$).

(f) $\int \frac{x^2+x+1}{(x^2+1)^2}\,\mathrm{d}x$

Note that $\frac{x^2+x+1}{(x^2+1)^2} = \frac{x^2+1}{(x^2+1)^2} + \frac{x}{(x^2+1)^2} = \frac{1}{(x^2+1)} + \frac{x}{(x^2+1)^2}$ so we don't need to do partial fractions.

Alternatively, $\frac{x^2+x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$
$x^2 + x + 1 = (Ax + B)(x^2 + 1) + Cx + D = Ax^3 + Bx^2 + (A + C)x + B + D$
Note that $B + D = 1$, $A + C = 1$, $B = 1$ and $A = 0$. Hence, $D = 0$ and $C = 1$.

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,\mathrm{d}x = \int \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\,\mathrm{d}x = \int \frac{1}{x^2+1}\,\mathrm{d}x + \int \frac{x}{(x^2+1)^2}\,\mathrm{d}x$$

To find the second integral, let $u = x^2 + 1$, $\mathrm{d}u = 2x\,\mathrm{d}x$, that is $x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$. Hence,

$$\int \frac{1}{x^2+1}\,\mathrm{d}x + \int \frac{x}{(x^2+1)^2}\,\mathrm{d}x = \tan^{-1}x + \frac{1}{2}\int \frac{1}{u^2}\,\mathrm{d}u = \tan^{-1}x - \frac{1}{2}\cdot\frac{1}{u} + C$$
$$= \boxed{\tan^{-1}x - \frac{1}{2(x^2+1)} + K}$$

(g) $\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x$

$\frac{x^3+6x-2}{x^4+6x^2} = \frac{x^3+6x-2}{x^2(x^2+6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+6}$

$$x^3 + 6x - 2 = Ax(x^2+6) + B(x^2+6) + (Cx+D)x^2$$
$$= Ax^3 + 6Ax + Bx^2 + 6B + Cx^3 + Dx^2$$
$$= (A+C)x^3 + (B+D)x^2 + 6Ax + 6B(1)$$

Note that $6B = -2$, $6A = 6$, $B + D = 0$, and $A + C = 1$. Hence, $B = -\frac{1}{3}$. $A = 1$, $D = \frac{1}{3}$, and $C = 0$.

$$\int \tfrac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x = \int \tfrac{1}{x} - \tfrac{\frac{1}{3}}{x^2} + \tfrac{\frac{1}{3}}{x^2+6}\,\mathrm{d}x = \int \tfrac{1}{x}\,\mathrm{d}x - \tfrac{1}{3}\int x^{-2}\,\mathrm{d}x + \tfrac{1}{3}\int \tfrac{1}{x^2+6}\,\mathrm{d}x$$

$$= \boxed{\ln|x| + \frac{1}{3x} + \frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + K}$$

7.5 Strategy for Integration

Strategy

1. Simplify the integrand if possible
   - **Algebraic manipulation:** For example, $\int \sqrt{x}(1+\sqrt{x})\,\mathrm{d}x = \int (x^{\frac{1}{2}}+x)\,\mathrm{d}x$
   - **Trigonometric identities:** For example, $\int (\sin x+\cos x)^2\,\mathrm{d}x = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\,\mathrm{d}x = \int (1+2\sin x\cos x)\,\mathrm{d}x$.
2. Look for an "obvious" substitution
   - Try to find some function in the integrand whose differential also occurs, apart from a constant factor. For example, $\int \frac{x}{x^2-1}\,\mathrm{d}x$, $u = x^2-1$ and $\mathrm{d}u = 2x\,\mathrm{d}x$.
   - This way we avoid the method of partial fractions.
3. Classify the integrand according to its form.
   (a) **Trigonometric functions:** If $f(x)$ is a product of powers of $\sin x/\cos x$, $\tan x/\sec x$, $\cot x/\csc x$ then use substitutions from section 7.2 and trigonometric identities.
   (b) **Rational functions:** If $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, then apply methods of partial fractions.
   (c) **Integration by parts:** If $f(x)$ is the product of a polynomial or a power of x and a trigonometric/exponential/logarithmic function, then try integration by parts. Choose u and $\mathrm{d}v$ according to the advice given in section 7.1
   (d) **Radicals:** If $\sqrt{\pm x^2 \pm a^2}$ occurs, use a trigonometric substitution according to the table in section 7.3.
4. If everything else fails...
   - Manipulate the integrand. For example, $\int \frac{1}{1-\cos x}\,\mathrm{d}x = \int \frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\,\mathrm{d}x$
   - Try a not so obvious substitution.

You might need the following integrals:

- $\boxed{\int \sec x\,\mathrm{d}x = \ln|\sec x+\tan x|}$
- $\boxed{\int \csc x\,\mathrm{d}x = -\ln|\csc x+\cot x|}$
- $\boxed{\int \tan x\,\mathrm{d}x = \ln|\sec x|}$
- $\boxed{\int \cot x\,\mathrm{d}x = \ln|\sin x|}$
- $\boxed{\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$
- $\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right),\ a>0}$

Example. Evaluate $\int \frac{x+2}{x^2+3x-4}\,dx$.

Partial fractions!

$$\frac{x+2}{x^2+3x-4} = \frac{x+2}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1} \quad \left]\,(x+4)(x-1)\right.$$

$$x+2 = A(x-1) + B(x+4)$$

$x=1$: $3 = B(5) \Rightarrow B = \frac{3}{5}$

$x=-4$: $-2 = A(-5) \Rightarrow A = \frac{2}{5}$

$$\int \frac{x+2}{x^2+3x-4}\,dx = \frac{2}{5}\int \frac{1}{x+4}\,dx + \frac{3}{5}\int \frac{1}{x-1}\,dx$$

$$= \boxed{\frac{2}{5}\ln|x+4| + \frac{3}{5}\ln|x-1| + C}$$

Example. Evaluate $\int \frac{\sin x}{\cos^3 x}\,dx$.

$u = \cos x$

$du = -\sin x\,dx$

$$\int \frac{\sin x}{\cos^3 x}\,dx = -\int \frac{1}{u^3}\,du = -\int u^{-3}\,du = -\frac{u^{-2}}{-2} + C = \frac{1}{2}\cdot\frac{1}{u^2} + C$$

$$= \boxed{\frac{1}{2}\cdot\frac{1}{\cos^2 x} + C} = \frac{1}{2}\sec^2 x + C$$

Example. Evaluate $\int \frac{\sin^3 x}{\cos x}\,dx$.

$$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x \cdot \sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\cdot \sin x}{\cos x}\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\int \frac{1-u^2}{u}\,du = -\int \frac{1}{u} - u\,du$$

$$= -\ln|u| + \frac{u^2}{2} + C$$

$$= \boxed{-\ln|\cos x| + \frac{\cos^2 x}{2} + C}$$

Example. Evaluate $\int x \sec x \tan x\,dx$.

ILATE

Integration by parts!

$u = x \quad dv = \sec x \tan x\,dx$

$du = dx \quad v = \int \sec x \tan x\,dx = \sec x$

$$\int x \sec x \tan x\,dx = x\sec x - \int \sec x\,dx$$

$$= \boxed{x\sec x - \ln|\sec x + \tan x| + C}$$

Example. Evaluate $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Trig. Substitution

$x = a\sin\theta = \sin\theta$

$dx = \cos\theta\,d\theta$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta\cdot\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = \int \frac{\sin^2\theta\cdot\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int \frac{\sin^2\theta\cdot\cos\theta}{\cos\theta}\,d\theta$$

$$= \int \sin^2\theta\,d\theta = \frac{1}{2}\int 1-\cos 2\theta\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta \Rightarrow \sin 2\theta = 2\sin\theta\cos\theta$

$x = \sin\theta$

$\sin^{-1} x = \theta$

$$= \frac{1}{2}\sin^{-1} x - \frac{1}{4}\cdot 2\sin\theta\cos\theta + C$$

$$= \boxed{\frac{1}{2}\sin^{-1} x - \frac{1}{2}\cdot x\cdot\sqrt{1-x^2} + C}$$

Example. Evaluate $\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$.

$u = \sqrt{t}$

$du = \frac{1}{2\sqrt{t}}\,dt$

$2\,du = \frac{1}{\sqrt{t}}\,dt$

$$\int \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt = 2\int e^u\,du = 2e^u + C$$

$$= \boxed{2e^{\sqrt{t}} + C}$$

Example. Evaluate $\int \tan^3 x \sec^4 x \, dx$.

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \cdot \sec^2 x \, dx = \int \tan^3 x (\tan^2 x + 1) \cdot \sec^2 x \, dx$$

$u = \tan x$, $du = \sec^2 x \, dx$

$$= \int u^3 (u^2 + 1) \, du = \int u^5 + u^3 \, du = \frac{u^6}{6} + \frac{u^4}{4} + C = \boxed{\frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C}$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^2 x \sec^3 x \cdot \sec x \tan x \, dx = \int (\sec^2 x - 1) \cdot \sec^3 x \cdot \sec x \tan x \, dx$$

$u = \sec x$, $du = \sec x \tan x \, dx$

$$= \int (u^2 - 1) u^3 \, du = \int u^5 - u^3 \, du = \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \boxed{\frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C}$$

Example. Evaluate $\int \frac{1}{x\sqrt{4x^2+1}} \, dx$.

$2x = \tan\theta \Rightarrow x = \frac{1}{2} \cdot \tan\theta$

$dx = \frac{1}{2} \sec^2\theta \, d\theta$

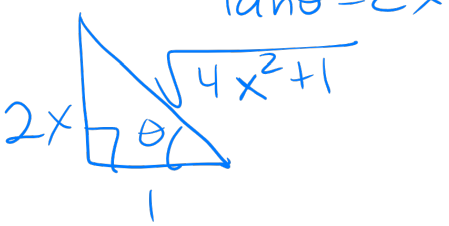
$$\int \frac{1}{x\sqrt{4x^2+1}} \, dx = \int \frac{\frac{1}{2}\sec^2\theta \, d\theta}{\frac{1}{2}\tan\theta \sqrt{\tan^2\theta + 1}} = \int \frac{\sec^2\theta}{\tan\theta \sqrt{\sec^2\theta}} \, d\theta$$

$$= \int \frac{\sec\theta}{\tan\theta} \, d\theta = \int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} \, d\theta = \int \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} \, d\theta$$

$$= \int \frac{1}{\sin\theta} \, d\theta = \int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + C$$

$$= \boxed{-\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C}$$

$\tan\theta = 2x$



Example. Evaluate $\int \frac{2x-3}{x^3+3x}\,dx$.

$$\left[\frac{2x-3}{x^3+3x} = \frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}\right] \cdot x(x^2+3)$$

$$2x-3 = A(x^2+3) + (Bx+C)\cdot x$$
$$= Ax^2 + 3A + Bx^2 + Cx$$
$$= (A+B)x^2 + Cx + 3A$$

$3A = -3$

$A = -1$

$C = 2$

$A+B=0 \Rightarrow B = -A = -(-1) = 1$

$$\int \frac{2x-3}{x^3+3x}\,dx = \int \frac{-1}{x} + \frac{x+2}{x^2+3}\,dx = -\int \frac{1}{x}\,dx + \int \frac{x}{x^2+3}\,dx$$

$$+ 2\int \frac{1}{x^2+3}\,dx$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= -\ln|x| + \frac{1}{2}\int \frac{1}{u}\,du + 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$= -\ln|x| + \frac{1}{2}\ln|u| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$= \boxed{-\ln|x| + \frac{1}{2}\ln|x^2+3| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

Homework 7.5 Solutions: Strategy for Integration

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the integral.

   (a) $\int \frac{\sin^3 x}{\cos x}\,dx$

   $$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x \sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx$$

   Let $u = \cos x$, then $du = -\sin x\,dx$. Hence,

   $$\int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = -\int \frac{(1-u^2)}{u}\,du = -\int \frac{1}{u} - u\,du = -\left[\ln|u| - \frac{u^2}{2}\right] + C$$

   $$= \boxed{-\ln|\cos x| + \frac{\cos^2 x}{2} + C}$$

   (b) $\int t \sin t \cos t\,dt$

   Note that $\sin t \cos t = \frac{1}{2}\sin 2t$. Then,

   $$\int t \sin t \cos t\,dt = \frac{1}{2}\int t \sin 2t\,dt$$

   Let $u = t$ and $dv = \sin 2t\,dt$, then $du = dt$ and $v = -\frac{1}{2}\cos 2t$. Hence,

   $$\frac{1}{2}\int t \sin 2t\,dt = \frac{1}{2}\left[-\frac{1}{2}t\cos 2t + \int \frac{1}{2}\cos 2t\,dt\right] = \boxed{-\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t + C}$$

   (c) $\int \frac{2x-3}{x^3+3x}\,dx$

   $\frac{2x-3}{x^3+3x} = \frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$

   $2x - 3 = A(x^2+3) + (Bx+C)x = (A+B)x^2 + Cx + 3A$

   Note that $3A = -3$, $C = 2$ and $A + B = 0$. Hence, $A = -1$ and $B = 1$. Hence,

$$\int \tfrac{2x-3}{x^3+3x}\,\mathrm{d}x = \int \Big(-\tfrac{1}{x} + \tfrac{x+2}{x^2+3} \Big)\,\mathrm{d}x = -\int \tfrac{1}{x}\,\mathrm{d}x + \int \tfrac{x}{x^2+3}\,\mathrm{d}x + \int \tfrac{2}{x^2+3}\,\mathrm{d}x$$

$$= -\ln|x| + \tfrac{1}{2}\int \tfrac{1}{u}\,\mathrm{d}u + \tfrac{2}{\sqrt{3}}\tan^{-1}\tfrac{x}{\sqrt{3}} + C \qquad (u = x^2+3)$$

$$= \boxed{-\ln|x| + \frac{1}{2}\ln|x^2+3| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C}$$

(d) $\int \ln(1+x^2)\,\mathrm{d}x$

Let $u = \ln(1+x^2)$ and $\mathrm{d}v = \mathrm{d}x$. Then, $\mathrm{d}u = \frac{2x}{1+x^2}\,\mathrm{d}x$ and $v = x$. Then,

$$\int \ln(1+x^2)\,\mathrm{d}x = x\ln(1+x^2) - \int \tfrac{2x^2}{1+x^2}\,\mathrm{d}x = x\ln(1+x^2) - \int 2 - \tfrac{2}{1+x^2}\,\mathrm{d}x$$

$$= \boxed{x\ln(1+x^2) - 2x + 2\tan^{-1}x + C}$$

(e) $\int x\sec x\tan x\,\mathrm{d}x$

Let $u = x$ and $\mathrm{d}v = \sec x\tan x\,\mathrm{d}x$. Then, $\mathrm{d}u = \mathrm{d}x$ and $v = \sec x$. Hence,

$$\int x\sec x\tan x\,\mathrm{d}x = x\sec x - \int \sec x\,\mathrm{d}x = \boxed{x\sec x - \ln|\sec x + \tan x| + C}$$

(f) $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$

Let $x = \sin\theta$, then $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. If $x = 0$, then $\theta = 0$ and if $x = \frac{\sqrt{2}}{2}$, then $\theta = \frac{\pi}{4}$. Hence,

$$\int_0^{\frac{\sqrt{2}}{2}} \tfrac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \tfrac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \tfrac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \sin^2\theta\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{2}\Big(1-\cos 2\theta\Big)\,\mathrm{d}\theta = \tfrac{1}{2}\Big[\theta - \tfrac{\sin 2\theta}{2}\Big]_0^{\frac{\pi}{4}} = \tfrac{1}{2}\Big(\tfrac{\pi}{4} - \tfrac{\sin\frac{\pi}{2}}{2}\Big) = \boxed{\frac{\pi}{8} - \frac{1}{4}}$$

(g) $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t$

Let $u = \sqrt{t}$, then $\mathrm{d}u = \frac{1}{2\sqrt{t}}\,\mathrm{d}t$, which implies $\frac{1}{\sqrt{t}}\,\mathrm{d}t = 2\,\mathrm{d}u$. If $t = 1$, then $u = \sqrt{1} = 1$ and if $t = 4$, then $u = \sqrt{4} = 2$. Hence,

$$\int_1^4 \tfrac{e^{\sqrt{t}}}{\sqrt{t}}\,\mathrm{d}t = \int_1^2 e^u\,\mathrm{d}u = 2\Big[e^u\Big]_1^2 = \boxed{2(e^2 - e)}$$

(h) $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,\mathrm{d}x$

Let $u = 1 + (\ln x)^2$, then $\mathrm{d}u = 2(\ln x)\cdot\frac{1}{x}\,\mathrm{d}x$. Hence,

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u = \frac{1}{2}\int u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{1}{2}\left[2u^{\frac{1}{2}}\right] + C = \boxed{\sqrt{1+(\ln x)^2} + C}$$

(i) $\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,\mathrm{d}x$

$$\frac{3x^2+1}{x^3+x^2+x+1} = \frac{3x^2+1}{x^2(x+1)+(x+1)} = \frac{3x^2+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$3x^2+1 = A(x^2+1) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + A + C$$

Note that $A+C=1$, $B+C=0$, and $A+B=3$, which implies $B=-C=3-A$, hence $A+A-3=1$ and $A=2$. Thus, $B=1$ and $C=-1$. Therefore,

$$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,\mathrm{d}x = \int_0^1 \frac{2}{x+1} + \frac{x-1}{x^2+1}\,\mathrm{d}x = \int_0^1 \frac{2}{x+1}\,\mathrm{d}x + \int_0^1 \frac{x}{x^2+1}\,\mathrm{d}x - \int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x$$

$$= 2\int_0^1 \frac{1}{x+1}\,\mathrm{d}x + \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{u}\,\mathrm{d}u - \int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x \qquad (u = x^2+1)$$

$$= \left[2\ln|x+1| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x\right]_0^1 = 2\ln 2 + \frac{1}{2}\ln 2 - \tan^{-1}(1) - \Big(2\ln 1 + \frac{1}{2}\ln 1 - \tan^{-1}(0)\Big) = \frac{5}{2}\ln 2 - \frac{\pi}{4} - (0+0-0) = \boxed{\frac{5}{2}\ln 2 - \frac{\pi}{4}}$$

(j) $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\,\mathrm{d}x$

Let $u = \frac{3}{x}$, then $\mathrm{d}u = -\frac{3}{x^2}\,\mathrm{d}x$, which implies $-\frac{1}{3}\,\mathrm{d}u = \frac{1}{x^2}\,\mathrm{d}x$

$$\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\,\mathrm{d}x = -\frac{1}{3}\int_3^1 e^u\,\mathrm{d}u = \frac{1}{3}\int_1^3 e^u\,\mathrm{d}u = \frac{1}{3}\Big[e^u\Big]_1^3 = \boxed{\frac{1}{3}(e^3 - e)}$$

(k) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin\theta\cot\theta}{\sec\theta}\,\mathrm{d}\theta$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin\theta\cot\theta}{\sec\theta}\,\mathrm{d}\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin\theta\cdot\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}}\,\mathrm{d}\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\theta\,\mathrm{d}\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2}\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} - \frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) = \boxed{\frac{\pi}{12}}$$

(l) $\int \frac{(x-1)e^x}{x^2}\,\mathrm{d}x$

Let $u = (x-1)e^x$ and $\mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x$. Then, $\mathrm{d}u = [e^x + (x-1)e^x]\,\mathrm{d}x = xe^x\,\mathrm{d}x$ and $v = -\frac{1}{x}$. Hence,

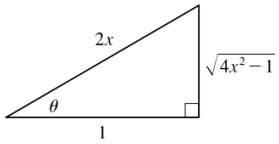
$$\int \frac{(x-1)e^x}{x^2}\,\mathrm{d}x = -\frac{(x-1)e^x}{x} + \int e^x\,\mathrm{d}x = -\frac{xe^x}{x} + \frac{e^x}{x} + e^x + C = \boxed{\frac{e^x}{x} + C}$$

(m) $\int \frac{1}{x^2\sqrt{4x^2-1}}\,\mathrm{d}x$

Let $2x = \sec\theta$, then $x = \frac{\sec\theta}{2}$ and $\mathrm{d}x = \frac{1}{2}\sec\theta\tan\theta\,\mathrm{d}\theta$. Hence,

$$\int \frac{1}{x^2\sqrt{4x^2-1}}\,\mathrm{d}x = \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\frac{\sec^2\theta}{4}\sqrt{\sec^2\theta-1}}\,\mathrm{d}\theta = 4\cdot\frac{1}{2}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\,\mathrm{d}\theta$$

$$= 2\int \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}\,\mathrm{d}\theta = 2\int \frac{1}{\sec\theta}\,\mathrm{d}\theta = 2\int \cos\theta\,\mathrm{d}\theta = 2\sin\theta + C = \boxed{\frac{\sqrt{4x^2-1}}{x} + C}$$



(n) $\int \sin^5 t\cos^4 t\,\mathrm{d}t$

$$\int \sin^5 t\cos^4 t\,\mathrm{d}t = \int \sin^4 t\cos^4 t\sin t\,\mathrm{d}t = \int (1-\cos^2 t)^2\cos^4 t\sin t\,\mathrm{d}t$$

Let $u = \cos t$, then $\mathrm{d}u = -\sin t\,\mathrm{d}t$. Hence,

$$\int (1-\cos^2 t)^2\cos^4 t\sin t\,\mathrm{d}t = -\int (1-u^2)^2\,\mathrm{d}u = -\int (u^4 - 2u^6 + u^8)\,\mathrm{d}u$$

$$= -\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right] + C = \boxed{-\frac{\cos^5 t}{5} + \frac{2\cos^7 t}{7} - \frac{\cos^9 t}{9} + C}$$

7.6 Improper Integrals

So far we have been working with integrals of the form $\int_a^b f(x)\,dx$. Note:

- f is continuous on [a,b]
- [a,b] is a finite interval

In this section we consider:

- f has a discontinuity on [a,b]
- integration over an infinite interval

In either case the integral is called an **improper integral**.

Type 1: Infinite Intervals

- $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided this limit exists as a finite number.
- $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$ provided this limit exists as a finite number.
- If the limit exists, then $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent**. Otherwise they are called **divergent**.
- If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

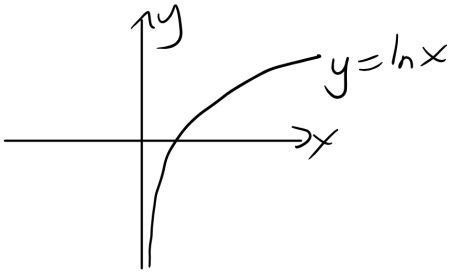
for any real number a.

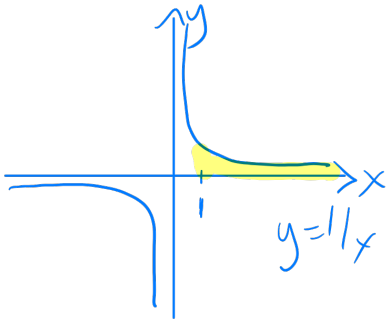
Example. Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}\Big[\ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty}\Big[\ln|t| - \ln 1\Big] = \infty$$

Divergent





Example. Determine whether the integral $\int_3^\infty \frac{1}{(x-2)^{\frac{3}{2}}}\,dx$ is convergent or divergent.

$$\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty}\int_1^{t-2} \frac{1}{u^{3/2}}\,du$$

$u = x-2$, $du = dx$

$$= \lim_{t\to\infty}\left[\frac{u^{-1/2}}{-1/2}\right]_1^{t-2} = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{u}}\right]_1^{t-2} = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right]$$

$$= 0+2 = \boxed{2} \quad \text{convergent}$$

Example. Determine whether the integral $\int_1^\infty \frac{1}{x^2+x}\,dx$ is convergent or divergent.

$$\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x} - \frac{1}{x+1}\,dx$$

$$\left[\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}\right]\cdot x(x+1)$$

$$1 = A(x+1) + Bx$$

$$= (A+B)x + A$$

$A = 1 \qquad A+B = 0 \Rightarrow B = -1$

$$= \lim_{t\to\infty}\left[\ln|x| - \ln|x+1|\right]_1^t$$

$$= \lim_{t\to\infty}\left[\underbrace{\ln|t| - \ln|t+1|}_{\infty-\infty} - \ln 1 + \ln 2\right]$$

$$= \lim_{t\to\infty}\left[\ln\left|\frac{t}{t+1}\right| - 0 + \ln 2\right] = 0 + \ln 2 = \boxed{\ln 2}$$

$\frac{\infty}{\infty}$

convergent

$$\lim_{t\to\infty} \ln\left|\frac{t}{t+1}\right| \overset{H}{=} \lim_{t\to\infty} \ln\left|\frac{1}{1+0}\right| = \ln 1 = 0$$

Example. Evaluate $\int_{-\infty}^{0} xe^x \, dx$

$$\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \int_t^0 xe^x dx = \lim_{t\to-\infty} \left[xe^x - e^x\right]_t^0$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= \lim_{t\to-\infty} \left[0e^0 - e^0 - te^t + e^t\right] = 0 - 1 - 0 + 0 = \boxed{-1}$$

($te^t \to -\infty \cdot 0$, $e^t \to 0$)

convergent

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}} \overset{H}{=} \lim_{t\to-\infty} \frac{1}{-e^{-t}} = 0$$

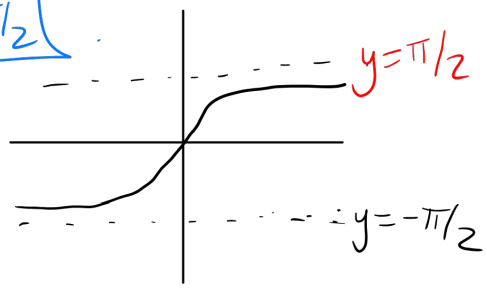
($\frac{t}{e^{-t}} \to \frac{\infty}{\infty}$)

Example. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2} dx = \lim_{t\to-\infty} \left[\tan^{-1} x\right]_t^0$$

$$= \lim_{t\to-\infty} \left[\tan^{-1} 0 - \tan^{-1} t\right] = 0 + \pi/2 = \boxed{\pi/2}$$



$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx$$

$$= \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t = \lim_{t\to\infty} \left[\tan^{-1} t - \tan^{-1} 0\right] = \boxed{\pi/2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi/2 + \pi/2 = \boxed{\pi} \quad \text{convergent}$$

Example. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

If $p=1$, $\int_1^\infty \frac{1}{x}\,dx$ is divergent

$p\neq 1$: $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$

$= \lim_{t\to\infty}\left[\frac{t^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right]$ $\Rightarrow$ $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p>1$

If $1-p>0 \Rightarrow t^{1-p}\to\infty$

$1-p<0 \Rightarrow t^{1-p}\to 0$

Type 2: Discontinuous Integrands

- If f is continuous on $[a, b)$ and is discontinuous at b, then $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$ if this limit exists as a finite number.
- If f is continuous on $(a, b]$ and is discontinuous at a, then $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$ if this limit exists as a finite number.
- The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.
- If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example. Find $\int_3^6 \frac{1}{\sqrt{x-3}}\,dx$.

Notice $f(x) = \frac{1}{\sqrt{x-3}}$ is discontinuous at $x=3$

$$\int_3^6 \frac{1}{\sqrt{x-3}}\,dx = \lim_{t\to 3^+} \int_t^6 \frac{1}{\sqrt{x-3}}\,dx = \lim_{t\to 3^+} \int_{t-3}^{3} u^{-1/2}\,du$$

$u = x-3$, $du = dx$

$$= \lim_{t\to 3^+} \left[\frac{u^{1/2}}{1/2}\right]_{t-3}^{3} = \lim_{t\to 3^+} \left[2\sqrt{u}\right]_{t-3}^{3} = \lim_{t\to 3^+} \left[2\sqrt{3} - 2\sqrt{t-3}\right]$$

$$= \boxed{2\sqrt{3}} \quad \text{convergent}$$

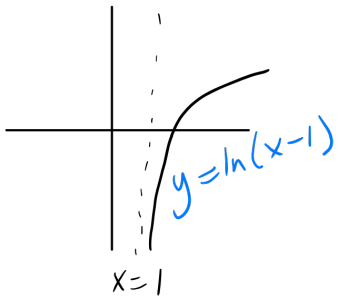
Example. Find $\int_0^2 \frac{1}{x-1}\,dx$.

$\frac{1}{x-1}$ is discontinuous at $x=1$

$$\int_0^2 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx$$

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \left[\ln|x-1|\right]_0^t$$

$$= \lim_{t\to 1^-} \left[\ln|t-1| - \underbrace{\ln|-1|}_{=0}\right] = -\infty \quad \text{Divergent}$$



Example. Find $\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\,dx$.

$\frac{1}{\sqrt[4]{x+2}}$ is discontinuous at $x=-2$

$$\int_{-2}^{14} \frac{1}{\sqrt[4]{x+2}}\,dx = \lim_{t\to -2^+} \int_t^{14} (x+2)^{-1/4}\,dx = \lim_{t\to -2^+} \int_{t+2}^{16} u^{-1/4}\,du$$

($u = x+2$, $du = dx$)

$$= \lim_{t\to -2^+} \left[\frac{u^{3/4}}{3/4}\right]_{t+2}^{16} = \lim_{t\to -2^+} \left[\frac{4}{3}u^{3/4}\right]_{t+2}^{16} = \lim_{t\to -2^+} \left[\frac{4}{3}16^{3/4} - \frac{4}{3}(t+2)^{3/4}\right]$$

($(t+2)^{3/4} \to 0$)

$$= \frac{4}{3}\cdot 8 = \boxed{\frac{32}{3}}$$

Example. Find $\int_{-2}^{3} \frac{1}{x^4}\,dx$.

$\frac{1}{x^4}$ is discontinuous at $x=0$

$$\int_{-2}^{3} \frac{1}{x^4}\,dx = \int_{-2}^{0} \frac{1}{x^4}\,dx + \int_0^3 \frac{1}{x^4}\,dx$$

$$\int_{-2}^{0} \frac{1}{x^4}\,dx = \lim_{t\to 0^-} \int_{-2}^{t} x^{-4}\,dx = \lim_{t\to 0^-} \left[\frac{x^{-3}}{-3}\right]_{-2}^{t} = \lim_{t\to 0^-} \left[\frac{-1}{3t^3} + \frac{1}{3\cdot(-2)^3}\right]$$

$=$ D.N.E. Divergent

Example. Find $\int_0^1 \ln x\,dx$.

$\ln 0$ is undefined $\Rightarrow \ln x$ is discontinuous at $x=0$

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \lim_{t\to 0^+} \left[x\ln x - x\right]_t^1$$

(done in 7.1)

$$= \lim_{t\to 0^+} \left[\underbrace{1\cdot\ln 1}_{=0} - 1 - (t\ln t - t)\right] = \boxed{-1} \text{ convergent}$$

($t\to 0$, $\ln t \to -\infty$)

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{H}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-1/t^2} = \lim_{t\to 0^+} \frac{1}{t}\cdot(-t^2) = \lim_{t\to 0^+} (-t) = 0$$

($\frac{1}{t} \to -\infty$, $-1/t^2 \to -\infty$)

Homework 7.6 Solutions: Improper integrals

Prof. Leyda Almodóvar

Name: _____

1. Explain why each of the following integrals is improper.

   (a) $\int_0^\infty \frac{1}{1+x^3}\,dx$

   The integral has an infinite interval of integration. It is an improper integral of Type 1.

   (b) $\int_0^{\frac{\pi}{4}} \cot x\,dx$

   The integrand $\cot x$ has an infinite discontinuity at $x = 0$. $\int_0^{\frac{\pi}{4}} \cot x\,dx$ is an improper integral of Type 2.

2. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

   (a) $\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

   $$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\left[\tfrac{4}{3}(x+1)^{\frac{3}{4}}\right]_0^t = \lim_{t\to\infty}\tfrac{4}{3}[(t+1)^{\frac{3}{4}} - 1] = \infty$$

   Hence, the integral $\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dt$ is divergent.

   (b) $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

   $$\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx$$

   Let $u = 2x+1$, then $du = 2\,dx$, which implies $dx = \frac{1}{2}\,du$. If $x = 1$, then $u = 3$ and if $x = t$, then $u = 2t+1$. Hence,

   $$\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\tfrac{1}{2}\int_3^{2t+1}\frac{1}{u^3}\,du = \lim_{t\to\infty}\left[-\tfrac{1}{4}u^{-2}\right]_3^{2t+1}$$

   $$= -\tfrac{1}{4}\lim_{t\to\infty}\left[\tfrac{1}{(2t+1)^2} - \tfrac{1}{9}\right] = -\tfrac{1}{4}(0 - \tfrac{1}{9}) = \boxed{\frac{1}{36}}$$

Hence, the integral $\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x$ is convergent.

(c) $\int_{-\infty}^0 2^r\,\mathrm{d}r$

$$\int_{-\infty}^0 2^r\,\mathrm{d}r = \lim_{t\to-\infty}\int_t^0 2^r\,\mathrm{d}r = \lim_{t\to-\infty}\left[\frac{2^r}{\ln 2}\right]_t^0 = \lim_{t\to-\infty}\left(\frac{2^0}{\ln 2}-\frac{2^t}{\ln 2}\right) = \frac{1}{\ln 2}-0 = \boxed{\frac{1}{\ln 2}}$$

Hence, the integral $\int_{-\infty}^0 2^r\,\mathrm{d}r$ is convergent.

(d) $\int_{-\infty}^\infty (y^3-3y^2)\,\mathrm{d}y$

$$\int_{-\infty}^\infty (y^3-3y^2)\,\mathrm{d}y = \int_{-\infty}^0 (y^3-3y^2)\,\mathrm{d}y + \int_0^\infty (y^3-3y^2)\,\mathrm{d}y$$

Note that if either of the integrals is divergent, the original integral is divergent as well.

$$\int_{-\infty}^0 (y^3-3y^2)\,\mathrm{d}y = \lim_{t\to-\infty}\int_t^0 (y^3-3y^2)\,\mathrm{d}y = \lim_{t\to-\infty}\left[\frac{y^4}{4}-y^3\right]_t^0$$

$$= \lim_{t\to-\infty}\left(\frac{t^4}{4}-t^3-0+0\right) = \infty$$

Hence, the integral $\int_{-\infty}^\infty (y^3-3y^2)\,\mathrm{d}y$ is divergent.

(e) $\int_1^\infty \frac{e^{-\frac{1}{x}}}{x^2}\,\mathrm{d}x$

Let $u=-\frac{1}{x}$, then $\mathrm{d}u=\frac{1}{x^2}\,\mathrm{d}x$. If $x=1$, then $u=-1$, and if $x=t$, then $u=-\frac{1}{t}$. Hence,

$$\int_1^\infty \frac{e^{-\frac{1}{x}}}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{e^{-\frac{1}{x}}}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_{-1}^{-\frac{1}{t}} e^u\,\mathrm{d}u = \lim_{t\to\infty}\left[e^u\right]_{-1}^{-\frac{1}{t}}$$

$$= \lim_{t\to\infty}\left(e^{-\frac{1}{t}}-e^{-1}\right) = \lim_{t\to\infty}\left(\frac{1}{e^{\frac{1}{t}}}-\frac{1}{e}\right) = \frac{1}{e^0}-\frac{1}{e} = \boxed{1-\frac{1}{e}}$$

Hence, the integral $\int_1^\infty \frac{e^{-\frac{1}{x}}}{x^2}\,\mathrm{d}x$ is convergent.

(f) $\int_0^\infty \sin\theta e^{\cos\theta}\,\mathrm{d}\theta$

Let $u=\cos\theta$, then $\mathrm{d}u=-\sin\theta\,\mathrm{d}\theta$. If $x=0$, $u=\cos 0=1$, and if $x=t$, then $u=\cos t$. Hence,

$$\int_0^\infty \sin\theta e^{\cos\theta}\,\mathrm{d}\theta = \lim_{t\to\infty}\int_0^t \sin\theta e^{\cos\theta}\,\mathrm{d}\theta = -\lim_{t\to\infty}\int_1^{\cos t} e^u\,\mathrm{d}u = -\lim_{t\to\infty}\left[e^{\cos t}-e^1\right]$$

Note that $\cos t$ oscillates between -1 and 1 as t approaches ∞. Hence, the limit does not exist and $\int_0^\infty \sin\theta e^{\cos\theta}\,\mathrm{d}\theta$ diverges.

(g) $\int_2^\infty \frac{1}{v^2+2v-3}\,\mathrm{d}v$

$\frac{1}{v^2+2v-3} = \frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$ implies $1 = A(v-1) + B(v+3)$

If $v = 1$, then $1 = A(0) + B(4)$, hence $B = \frac{1}{4}$.

If $v = -3$, then $1 = A(-4) + \frac{1}{4}(0)$, hence $A = -\frac{1}{4}$. Then,

$$\int_2^\infty \frac{1}{v^2+2v-3}\,\mathrm{d}v = \frac{1}{4}\lim_{t\to\infty}\int_2^t \frac{1}{v-1} - \frac{1}{v+3}\,\mathrm{d}v = \frac{1}{4}\lim_{t\to\infty}\Big[\ln|v-1| - \ln|v+3|\Big]_2^t$$

$$= \frac{1}{4}\Big(\ln|t-1| - \ln|t+3| - \ln 1 + \ln 5\Big) = \frac{1}{4}\Big(\ln\big|\tfrac{t-1}{t+3}\big| + \ln 5\Big)$$

$$= \frac{1}{4}\Big(\ln\big|1 - \tfrac{4}{t+3}\big| + \ln 5\Big) = \frac{1}{4}(\ln|1-0| + \ln 5) = \boxed{\frac{1}{4}\ln 5}$$

Hence, the integral $\int_2^\infty \frac{1}{v^2+2v-3}\,\mathrm{d}v$ is convergent.

(h) $\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x$

Let $u = \ln x$ and $\mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x$. Then, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = -\frac{1}{x}$. Then,

$$\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\left(\Big[-\frac{\ln x}{x}\Big]_1^t + \int_1^t \frac{1}{x^2}\,\mathrm{d}x\right) = \lim_{t\to\infty}\Big[\frac{\ln x}{x} - \frac{1}{x}\Big]_1^t =$$

$$\lim_{t\to\infty}\left(\frac{\ln t}{t} - \frac{1}{t} - \frac{\ln 1}{1} + \frac{1}{1}\right) = 0 - 0 - 0 + 1 = \boxed{1}$$

Note that $\lim_{t\to\infty}\frac{\ln t}{t} = \lim_{t\to\infty}\frac{\frac{1}{t}}{1} = 0$ by l'Hospital's rule.

Hence, the integral $\int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x$ is convergent.

(i) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$

Note that $\frac{1}{\sqrt{1-x^2}}$ is undefined when $x = 1$. Hence,

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \lim_{t\to 1^-}\Big[\sin^{-1}x\Big]_0^t = \lim_{t\to 1^-}(\sin^{-1}t - 0) = \boxed{\frac{\pi}{2}}$$

Hence, the integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ is convergent.

(j) $\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,\mathrm{d}x$

Note that $\frac{1}{\sqrt[3]{5-x}}$ is undefined when $x = 5$. Hence,

$$\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,\mathrm{d}x = \lim_{t\to 5^-}\int_0^t (5-x)^{-\frac{1}{3}}\,\mathrm{d}x = \lim_{t\to 5^-}\Big[-\tfrac{3}{2}(5-x)^{\frac{2}{3}}\Big]_0^t$$

$$= -\tfrac{3}{2}\lim_{t\to 5^-}\Big[(5-t)^{\frac{2}{3}} - (5-0)^{\frac{2}{3}}\Big] = 0 + \tfrac{3}{2}5^{\frac{2}{3}} = \boxed{\frac{3}{2}5^{\frac{2}{3}}}$$

(k) $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x$

Note that $\frac{1}{\sqrt[3]{x-1}}$ is discontinuous at $x = 1$. Then,

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x$$

First,

$$\int_0^1 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t (x-1)^{-\frac{1}{3}}\,\mathrm{d}x = \lim_{t\to 1^-}\Big[\tfrac{3}{2}(x-1)^{\frac{2}{3}}\Big]_0^t$$

$$= \tfrac{3}{2}\lim_{t\to 1^-}\Big[(t-1)^{\frac{2}{3}} - (-1)^{\frac{2}{3}}\Big] = \tfrac{3}{2}(0-1) = -\tfrac{3}{2}$$

Second,

$$\int_1^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = \lim_{t\to 1^+}\int_t^9 (x-1)^{-\frac{1}{3}}\,\mathrm{d}x = \lim_{t\to 1^+}\Big[\tfrac{3}{2}(x-1)^{\frac{2}{3}}\Big]_t^9$$

$$= \tfrac{3}{2}\lim_{t\to 1^+}\Big[(9-1)^{\frac{2}{3}} - (t-1)^{\frac{2}{3}}\Big] = \tfrac{3}{2}(8^{\frac{2}{3}} - (t-1)^{\frac{2}{3}}) = \tfrac{3}{2}(4-0) = 6$$

Thus,

$$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x = -\tfrac{3}{2} + 6 = \boxed{\frac{9}{2}}$$

Hence, the integral $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x$ is convergent.

8 Infinite Sequences and Series

8.1 Sequences

- A **sequence** is a list of numbers written in a definite order: $a_1, a_2, a_3, a_4, \ldots$.
- The list may be finite or infinite, but in this class we will only deal with infinite sequences.
- That means each term a_n will have a succesor a_{n+1}.
- **Notation:** $\{a_1, a_2, a_3, \ldots\}$, $\{a_n\}$, or $\{a_n\}_{n=1}^{\infty}$ → general formula for the nth term
- A sequence does not have to start at $n = 1$. → start at n=1 (it doesn't have to)

Example. Write down the first four terms of each of the following sequences.

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\frac{1+1}{1^2}, \frac{2+1}{2^2}, \frac{3+1}{3^2}, \frac{4+1}{4^2}, \ldots\right\}$

$= \left\{2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \ldots\right\}$

Alternating

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{\frac{(-1)^{0+1}}{2^0}, \frac{(-1)^{1+1}}{2^1}, \frac{(-1)^{2+1}}{2^2}, \frac{(-1)^{3+1}}{2^3}, \ldots\right\}$

$= \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots\right\}$

(c) $\left\{\sqrt{n-2}\right\}_{n=2}^{\infty} = \left\{\sqrt{2-2}, \sqrt{3-2}, \sqrt{4-2}, \sqrt{5-2}, \ldots\right\}$

$= \left\{\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\right\}$

$= \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots\right\}$

Goal: Study "end behavior" of a_n

Definition. A sequence $\{a_n\}$ has the **limit** L and we write $\lim_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n = L$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Example. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $\left\{\frac{n}{n+1}\right\}$ $a_n = \frac{n}{n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} \overset{\#}{=} \lim_{n\to\infty} \frac{1}{1+0} = \boxed{1}$ $\Rightarrow$ convergent

(b) $\{\sqrt{n}\}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \boxed{\infty}$ divergent

(c) $\{\sin n\}$

$\lim_{n\to\infty} \sin n = \text{D.N.E.}$ divergent

(d) $\left\{\sin\left(\frac{\pi}{n}\right)\right\}$

$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin 0 = \boxed{0}$ convergent

(e) $a_n = \dfrac{n^4}{n^3 - 2n}$

$\lim_{n\to\infty} \dfrac{n^4}{n^3-2n} = \lim_{n\to\infty} \dfrac{\frac{n^4}{n^3}}{\frac{n^3}{n^3} - \frac{2n}{n^3}} = \lim_{n\to\infty} \dfrac{n}{1 - \frac{2}{n^2}} \to 0 = \boxed{\infty}$ Divergent

(f) $a_n = e^{-\frac{1}{\sqrt{n}}}$

$\lim_{n\to\infty} e^{-1/\sqrt{n}} = e^0 = \boxed{1}$ convergent

(g) $\{n^2 e^{-n}\}$

$\lim_{n\to\infty} n^2 e^{-n}$ ($\infty \cdot 0$) $= \lim_{n\to\infty} \dfrac{n^2}{e^n}$ ($\frac{\infty}{\infty}$) $\overset{H}{=} \lim_{n\to\infty} \dfrac{2n}{e^n}$ ($\frac{\infty}{\infty}$) $\overset{H}{=} \lim_{n\to\infty} \dfrac{2}{e^n} = \boxed{0}$

convergent

(h) $a_n = \ln(2n^2 + 1) - \ln(n^2 + 1)$

$\lim_{n\to\infty} [\ln(2n^2+1) - \ln(n^2+1)]$ ($\infty - \infty$) $= \lim_{n\to\infty} \ln\left(\dfrac{2n^2+1}{n^2+1}\right)$ ($\frac{\infty}{\infty}$) $= \lim_{n\to\infty} \ln\left(\dfrac{\frac{2n^2}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}\right)$

$= \lim_{n\to\infty} \ln\left(\dfrac{2 + 1/n^2}{1 + 1/n^2}\right) \to 0 = \boxed{\ln 2}$ convergent

$\lim_{n\to\infty} r^n = \text{D.N.E.}$ if $r \neq 1$ or $-1 < r \leq 1$

Theorem. $\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$ and $\lim_{n\to\infty} r^n = 1$ if $r = 1$. The sequence $\{r^n\}$ is divergent for all other values of r.

Definition. A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \ldots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

Definition. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leq M$ for all $n \geq 1$. It is **bounded below** if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example. Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{-n^2\right\}_{n=0}^{\infty} = \{0, -1, -4, -9, -16, \ldots\}$

$a_{n+1} < a_n \Rightarrow$ decreasing $\Rightarrow$ monotonic

bounded above by 0 (or any positive number) } $\Rightarrow$ not bounded

not bounded below

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty} = \{1, -1, 1, -1, 1, -1, 1, \ldots\}$

not monotonic

bounded above by 1 } bounded

bounded below by -1

(c) $\left\{\frac{2}{n^2}\right\}_{n=3}^{\infty} = \left\{\frac{2}{9}, \frac{2}{16}, \frac{2}{25}, \frac{2}{36}, \cdots\right\}$

a_n is decreasing $\Rightarrow$ monotonic

bounded above by $\frac{2}{9}$ } bounded

bounded below by 0 }

Theorem. (Monotonic Sequence Theorem) Every bounded, monotonic sequence is convergent.

- The theorem does not say that if a sequence is not bounded and/or not monotonic that it is divergent.
- One way to show that a sequence is convergent is by showing that the sequence is bounded and monotonic.

Homework 8.1 Solutions: Sequences

Prof. Leyda Almodóvar

Name: _____

1. (a) What is a sequence?

   A sequence is a list of numbers written in a definite order.

   (b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?

   It means that we can make the terms a_n as close to 8 as we like by taking n sufficiently large.

   (c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

   It means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$.

2. List the first five terms of the sequence.

   (a) $a_n = \frac{n^2-1}{n^2+1}$

   $a_1 = \frac{1^2-1}{1^2+1} = 0$

   $a_2 = \frac{2^2-1}{2^2+1} = \frac{3}{5}$

   $a_3 = \frac{3^2-1}{3^2+1} = \frac{4}{5}$

   $a_4 = \frac{4^2-1}{4^2+1} = \frac{15}{17}$

   $a_5 = \frac{5^2-1}{5^2+1} = \frac{12}{13}$

   (b) $a_n = \cos(\frac{n\pi}{2})$

   $a_1 = \cos(\frac{\pi}{2}) = 0$

   $a_2 = \cos(\frac{2\pi}{2}) = \cos\pi = -1$

   $a_3 = \cos(\frac{3\pi}{2}) = 0$

   $a_4 = \cos(\frac{4\pi}{2}) = \cos 2\pi = 1$

   $a_5 = \cos(\frac{5\pi}{2}) = 0$

3. Determine whether the sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \frac{3+5n^2}{1+n}$

$$\lim_{n\to\infty} \frac{3+5n^2}{1+n} = \lim_{n\to\infty} \frac{\frac{3}{n}+\frac{5n^2}{n}}{\frac{1}{n}+\frac{n}{n}} = \lim_{n\to\infty} \frac{\frac{3}{n}+5n}{\frac{1}{n}+1} = \infty$$

Hence, the sequence $a_n = \frac{3+5n^2}{1+n}$ diverges.

(b) $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$

$$\lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{\frac{3\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}}+\frac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3}{1+\frac{2}{\sqrt{n}}} = \boxed{3}$$

Hence, the sequence $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$ converges.

(c) $a_n = \cos(\frac{n\pi}{n+1})$

$$\lim_{n\to\infty} \cos(\tfrac{n\pi}{n+1}) = \cos\left(\lim_{n\to\infty} \tfrac{n\pi}{n+1}\right) = \cos\left(\lim_{n\to\infty} \tfrac{\pi}{1+\frac{1}{n}}\right) = \cos\pi = \boxed{-1}$$

Hence, the sequence $a_n = \cos(\frac{n\pi}{n+1})$ converges.

(d) $a_n = e^{\frac{2n}{n+2}}$

$$\lim_{n\to\infty} e^{\frac{2n}{n+2}} = e^{\lim_{n\to\infty} \frac{2n}{n+2}} = e^{\lim_{n\to\infty} \frac{2}{1+\frac{2}{n}}} = \boxed{e^2}$$

Hence, the sequence $a_n = e^{\frac{2n}{n+2}}$ converges.

(e) $a_n = \ln(n+1) - \ln n$

Recall that $\ln(n+1) - \ln n = \ln\left(\frac{n+1}{n}\right)$. Hence,

$$\lim_{n\to\infty} \ln(n+1) - \ln n = \lim_{n\to\infty} \ln\left(\tfrac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\tfrac{1}{n}\right) = \ln(1+0) = \ln 1 = \boxed{0}$$

Hence, the sequence $a_n = \ln(n+1) - \ln n$ converges.

4. Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

   (a) $a_n = \cos n$

   This sequence is bounded since $-1 \leq \cos n \leq 1$ for all n. The sequence is not monotonic since cosine oscillates between -1 and 1.

   (b) $a_n = \frac{1-n}{2+n}$

   Note that a_n is decreasing since $\frac{1-n}{2+n} > \frac{1-(n+1)}{2+(n+1)}$ (the numerator is getting smaller and the denominator is getting larger as n increases). Hence, the sequence is monotonic.

   Since $a_1 = 0$, the sequence is bounded above by 0. Since $\lim_{n\to\infty} \frac{1-n}{2+n} = -1$, the sequence is bounded below by -1. Hence, the sequence is bounded.

   (c) $a_n = 2 + \frac{(-1)^n}{n}$

   Note that $a_n = 2+\frac{(-1)^n}{n} = \{1, \frac{5}{2}, \frac{5}{3}, \frac{9}{4}, \ldots\}$. Hence, the sequence is not monotonic and it is bounded since $1 \leq a_n \leq \frac{5}{2}$ for all n.

   (d) $a_n = n^3 - 3n + 3$

   Note that $\lim_{n\to\infty} (n^3 - 3n + 3) = \infty$ so the sequence is not bounded.

   To verify whether it is increasing or decreasing, let $f(x) = x^3 - 3x + 3$. Then, $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$. Note that $f'(x) > 0$ if $x > 1$. Hence, $f(x)$ is increasing, which implies a_n is increasing. Hence, $a_n = n^3-3n+3$ is monotonic.

8.2 Series

If we add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form $a_1 + a_2 + a_3 + \ldots$ which is called an **infinite series** (or just a **series**) and is denoted $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

→ sum starting at n=1

Example. Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$

Notice that

- $a_1 + a_2 = \frac{3}{4}$
- $a_1 + a_2 + a_3 = \frac{7}{8}$
- $a_1 + a_2 + a_3 + a_4 = \frac{15}{16}$
- $a_1 + a_2 + a_3 + a_4 + a_5 = \frac{31}{32}$
- $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = \frac{63}{64}$

Are the sums getting close to a number?

The sums are getting close to 1.

The first nth terms of the series

We consider **partial sums** $s_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i$.

These partial sums form a new sequence $\{s_n\} = \{s_1, s_2, s_3, \ldots, s_n\} = \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_n\}$ which may or may not have a limit.

Definition. Given a series $\sum_{n=1}^{\infty} a_n$ let $s_n = \sum_{i=1}^{n} a_i$ denote its nth partial sum. If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write $\sum_{n=1}^{\infty} a_n = s$.

Note:

- The number s is called the **sum** of the series.
- If the sequences $\{s_n\}$ is divergent, then the series is called **divergent**.

In other words, the sum of a series is the limit of the sequence of partial sums.

Example. Find $\sum_{n=1}^{\infty} a_n$ if the sum of the first n terms of the series is $s_n = \dfrac{2n}{3n+5}$.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} \overset{H}{=} \lim_{n\to\infty} \frac{2}{3+0} = \frac{2}{3}$

$\sum_{n=1}^{\infty} a_n = \boxed{\frac{2}{3}}$

Example. Consider a **geometric series** $a + ar + ar^2 + ar^3 + \ldots = \boxed{\sum_{n=1}^{\infty} ar^{n-1}}$, $a \neq 0$. Each term is obtained from the preceding one by multiplying it by a **common ratio** r.

Goal: Find a formula for s_n for any geometric series.

If $r=1$, $s_n = \underbrace{a+\cdots+a}_{n \text{ many}} = na \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} na = \infty$ or $-\infty$

$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ is divergent.

If $r \neq 1$, $s_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$rs_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$

$s_n - rs_n = a - ar^n$

$s_n(1-r) = a(1-r^n) \Rightarrow s_n = \dfrac{a(1-r^n)}{1-r}$

$\Rightarrow \lim_{n\to\infty} s_n = \dfrac{a}{1-r}$ if and only if $-1 < r < 1$

Recall: $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{D.N.E.} & \text{otherwise} \end{cases}$

Summary: The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges when $|r| < 1$ and its sum is $\dfrac{a}{1-r}$. Otherwise it diverges.

(a is the first term of the series)

Example. Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$

$r = \frac{20}{9} \div \left(-\frac{10}{3}\right) = \frac{20}{9} \cdot \left(-\frac{3}{10}\right) = -\frac{2}{3}$

Since $-1 < -2/3 < 1 \Rightarrow$ the geometric series converges to $\frac{a}{1-r} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{5/3} = \boxed{3}$

$\sum_{n=1}^{\infty} ar^{n-1}$

Example. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1}$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$r = 1/2 \Rightarrow -1 < 1/2 < 1 \Rightarrow$ The series converges to $\frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$

Example. Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent?

$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = 4 + \frac{16}{3} + \frac{64}{9} + \dots$

$\frac{64}{9} \div \frac{16}{3} = \frac{64}{9} \cdot \frac{3}{16} = \frac{4}{3}$

$\frac{16}{3} \div 4 = \frac{16}{3} \cdot \frac{1}{4} = 4/3$

$r = \frac{4}{3} > 1 \Rightarrow$ The geometric series diverges

$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3^{-(n-1)}$

$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{3}\right)^{n-1}$

Example. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

*This is NOT a geometric series.

$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$ ← Not clear what $\lim_{n\to\infty} S_n$ is

$\frac{1}{i(i+1)} = \frac{A}{i} + \frac{B}{i+1} \Rightarrow 1 = A(i+1) + Bi = Ai + Bi + A \Rightarrow A = 1$ and $A + B = 0 \Rightarrow B = -1$

$S_n = \sum_{i=1}^{n} \left[\frac{1}{i} - \frac{1}{i+1}\right] = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$+ \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = \boxed{1}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1}$$

Example. Test the series $\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$ for convergence. If it's convergent, find its sum.

*Not a geometric series

$S_n = \sum_{i=2}^{n} \frac{2}{i^2 - 1} \Rightarrow \frac{2}{(i-1)(i+1)} = \frac{A}{i-1} + \frac{B}{i+1} \Rightarrow 2 = A(i+1) + B(i-1) = (A+B)i + A - B$

$A + B = 0$, $A - B = 2$, $2A = 2 \Rightarrow A = 1$, $B = -1$

$S_n = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1}\right) = \left(1 - \frac{1}{3}\right) +$

$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots +$

$\left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$= 1 + 1/2 - \frac{1}{n} - \frac{1}{n+1} = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = 3/2$

$$\Rightarrow \boxed{\sum_{n=2}^{\infty} \frac{2}{n^2 - 1} = \frac{3}{2}}$$

Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exists or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

*If $\lim_{n\to\infty} a_n = 0$ we cannot reach any conclusions.

NOTE: With any series $\sum a_n$ we associate 2 sequences:

- The sequence $\{s_n\}$ of its partial sums.
- The sequence $\{a_n\}$ of its terms.

Example. Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{3n^2+7}$ diverges.

$a_n = \frac{n^2}{3n^2+7}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{3n^2+7} = \lim_{n\to\infty} \frac{1}{3+\frac{7}{n^2}} = \frac{1}{3} \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{3n^2+7}$ diverges by the Test for Divergence.

Example. Show that the series $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges.

$a_n = \frac{e^n}{n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^2} \overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2n} \overset{H}{=} \lim_{n\to\infty} \frac{e^n}{2} = \infty \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges by the Test for Divergence.

Homework 8.2 Solutions: Series

Prof. Leyda Almodóvar

Name: _____

1. (a) What is the difference between a sequence and a series?

   A sequence is an ordered list of numbers while a series is the *sum* of a list of numbers.

   (b) What is a convergent series? What is a divergent series?

   A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

   (c) What does it mean to say that $\sum_{n=1}^{\infty} a_n = 5$?

   It means that by adding sufficiently many terms of the series we can get as close as we like to the number 5.

2. Calculate the first eight terms of the sequence of partial sums. Does it appear that the series is convergent or divergent?

   (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

   $s_1 = a_1 = \frac{1}{\sqrt[3]{1}} = 1$
   $s_2 = a_1 + a_2 = 1 + \frac{1}{\sqrt[3]{2}}$
   $s_3 = s_2 + a_3 = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}}$
   $s_4 = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}}$
   $s_5 = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}}$
   $s_6 = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}}$
   $s_7 = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}} + \frac{1}{\sqrt[3]{7}}$
   $s_8 = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}} + \frac{1}{\sqrt[3]{7}} + \frac{1}{\sqrt[3]{8}} = \frac{3}{2} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{6}} + \frac{1}{\sqrt[3]{7}}$

   The series appears to be convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

Note that $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$

$s_1 = a_1 = \frac{1}{1!} = 1$

$s_2 = 1 - \frac{1}{2!} = \frac{1}{2}$

$s_3 = \frac{1}{2} + \frac{1}{3!} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$s_4 = \frac{2}{3} - \frac{1}{4!} = \frac{2}{3} - \frac{1}{24} = \frac{15}{24}$

$s_5 = \frac{15}{24} + \frac{1}{5!} = \frac{15}{24} + \frac{1}{120} = \frac{76}{120} = \frac{19}{30}$

$s_6 = \frac{19}{30} - \frac{1}{6!} = \frac{19}{30} - \frac{1}{720} = \frac{455}{720} = \frac{91}{144}$

$s_7 = \frac{91}{144} + \frac{1}{7!} = \frac{91}{144} + \frac{1}{5040} = \frac{3184}{5040} = \frac{199}{315}$

$s_8 = \frac{199}{315} - \frac{1}{8!} = \frac{199}{315} - \frac{1}{40320} = \frac{25471}{40320}$

The series appears to be convergent.

3. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $4 + 3 + \frac{9}{4} + \frac{27}{16} + \ldots$

Note that a is always the first term of a geometric series. Hence, $a = 4$. To find the common ratio r, we divide two consecutive terms. That is, $r = \frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, the series converges to $\frac{1}{1-r} = \frac{4}{1-\frac{3}{4}} = \frac{4}{\frac{1}{4}} = 16$. Notice that this series could be written as $\sum_{n=1}^{\infty} 4\left(\frac{3}{4}\right)^{n-1}$ or $\sum_{n=0}^{\infty} 4\left(\frac{3}{4}\right)^{n}$.

(b) $\sum_{n=1}^{\infty} \frac{5}{\pi^n}$

$\sum_{n=1}^{\infty} \frac{5}{\pi^n} = 5\sum_{n=1}^{\infty} \frac{1}{\pi^n} = 5\sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n$

This is a geometric series with $a = \frac{1}{\pi}$ and ratio $r = \frac{1}{\pi}$. Since, $|r| = \frac{1}{\pi} < 1$, the series $\sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n$ converges to $\frac{\frac{1}{\pi}}{1-\frac{1}{\pi}} = \frac{1}{\pi-1}$. Hence, the series $\sum_{n=1}^{\infty} \frac{5}{\pi^n}$ converges to $\frac{5}{\pi-1}$.

(c) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

$\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n} = \sum_{n=0}^{\infty} \frac{3^n \cdot 3}{(-2)^n} = 3\sum_{n=0}^{\infty} \left(\frac{3}{-2}\right)^n = 3\left(1 - \frac{3}{2} + \frac{9}{4} - \dots\right)$

This is a geometric series with $r = -\frac{3}{2}$. Since $|r| = \frac{3}{2} > 1$, the series diverges.

(d) $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

$\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n} \cdot 2^{-1}}{3^n} = \sum_{n=1}^{\infty} \frac{3 \cdot 2^{2n}}{3^n} = \sum_{n=1}^{\infty} \frac{3 \cdot (2^2)^n}{3^n} = 3\sum_{n=1}^{\infty} \frac{4^n}{3^n} = 3\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$

This is a geometric series with $r = \frac{4}{3}$. Since $|r| > 1$, the series diverges.

4. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\sum_{k=1}^{\infty} \frac{k^2}{k^2 - 2k + 5}$

$\lim_{k\to\infty} \frac{k^2}{k^2 - 2k + 5} = \lim_{k\to\infty} \frac{1}{1 - \frac{2}{k} + \frac{5}{k^2}} = 1 \neq 0$

Hence, the series $\sum_{k=1}^{\infty} \frac{k^2}{k^2 - 2k + 5}$ diverges.

(b) $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n}$

$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n} = \sum_{n=1}^{\infty} \frac{2^n}{e^n} + \sum_{n=1}^{\infty} \frac{4^n}{e^n}$

If either sum is divergent, the whole sum is divergent. Note that $\sum_{n=1}^{\infty} \frac{4^n}{e^n} = \sum_{n=1}^{\infty} \left(\frac{4}{e}\right)^n$ is a divergent geometric series since $r = \frac{4}{e} > 1$. Hence, the series $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n}$ is divergent.

(c) $\sum_{n=1}^{\infty} \frac{1}{1 + (\frac{2}{3})^n}$

First, Since $-1 < \frac{2}{3} < 1$, $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$. Then, $\lim_{n\to\infty} \frac{1}{1 + (\frac{2}{3})^n} = \frac{1}{1+0} = 1 \neq 0$

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{1 + (\frac{2}{3})^n}$ diverges by the Test for Divergence.

(d) $\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$

Note that $\sum_{k=0}^{\infty} (\sqrt{2})^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k$ is a geometric series with $a = \left(\frac{1}{\sqrt{2}}\right)^0 = 1$ and ratio $r = \frac{1}{\sqrt{2}}$. Since, $|r| < 1$, the series $\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$ converges to $\frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}$.

(e) $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

We apply L'Hospital's rule to $a_n = \frac{e^n}{n^2}$:

$\lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{e^n}{2n} = \lim_{n\to\infty} \frac{e^n}{2} = \infty \neq 0$

Hence, the series $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges by the Test for Divergence.

5. Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

(a) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

$\sum_{n=1}^{\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{\infty} [\ln n - \ln(n+1)]$. Hence,

$$s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \ldots + [\ln(n-1) - \ln n]$$
$$+[\ln n - \ln(n+1)] = \ln 1 - \ln(n+1) = -\ln(n+1)$$

Then, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n+1) = -\infty$

Hence, the series $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ is divergent.

(b) $\sum_{n=4}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$

$s_n = \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}}\right) + \ldots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{n+1}}$

Then, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{n+1}}\right) = \frac{1}{\sqrt{4}} - 0 = \frac{1}{2}$

Hence, the series $\sum_{n=4}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$ converges to $\frac{1}{2}$.

8.3 The Integral Test

- In general, it is difficult to find the exact sum of series.
- We found formulas for the *nth* partial sum s_n for geometric series and $\sum \frac{1}{n(n+1)}$, but usually it isn't easy to find such a formula.
- In the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example. Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

$a_n = \frac{1}{n^2+1} \Rightarrow f(x) = \frac{1}{x^2+1}$

- $f(x) = \frac{1}{x^2+1} > 0$ for all x
- f is a rational function and $x^2+1 \neq 0$ $\Rightarrow f$ is continuous
- The numerator is constant while the denominator increases as x increases $\Rightarrow f$ is decreasing

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t = \lim_{t\to\infty} \left[\tan^{-1} t - \tan^{-1} 1\right] = \pi/2 - \pi/4 = \pi/4$$

$\Rightarrow \int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

* Not necessarily to $\pi/4$

Note:

- We don't have to start at $n = 1$. For example, for $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$ we use $\int_4^{\infty} \frac{1}{(x-3)^2}\,dx$.
- f doesn't have to be always decreasing. We just need f to be *ultimately* decreasing, that is, decreasing for x larger than some number N.
- The value of the integral is not equal to the sum of the series: $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

$a_n = \frac{\ln n}{n} \Rightarrow f(x) = \frac{\ln x}{x}$

- f is continuous since $x \neq 0$ and $x > 1$
- $\ln x > 0$ if $x > 1 \Rightarrow f$ is positive on $(1, \infty)$

$f'(x) = \frac{(\ln x)' x - (x)' \cdot \ln x}{x^2}$

$= \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ when $1 - \ln x < 0 \Rightarrow 1 < \ln x$

$e < x$

$\Rightarrow f$ is ultimately decreasing

$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \int_0^{\ln t} u\,du = \lim_{t \to \infty} \left[\frac{u^2}{2}\right]_0^{\ln t}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \lim_{t \to \infty} \left[\frac{(\ln t)^2}{2} - \frac{0^2}{2}\right] = \infty \Rightarrow$ integral diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the Integral Test.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{5n-1}$ converges or diverges.

$a_n = \frac{2}{5n-1} \Rightarrow f(x) = \frac{2}{5x-1}$

- f is continuous since $x \neq \frac{1}{5}$
- f is positive since $5x-1 > 0$ on $(1, \infty)$
- f is decreasing since the numerator is constant and the den. increases as x increases.

$$\int_1^{\infty} \frac{2}{5x-1}\,dx = \lim_{t\to\infty} 2\int_1^t \frac{1}{5x-1}\,dx = \lim_{t\to\infty} \frac{2}{5}\int_4^{5t-1} \frac{1}{u}\,du$$

$u = 5x-1$

$du = 5\,dx$

$\frac{1}{5}du = dx$

$$= \lim_{t\to\infty} \frac{2}{5}\ln|u|\Big]_4^{5t-1} = \lim_{t\to\infty} \frac{2}{5}\left[\ln|5t-1| - \ln 4\right] = \infty$$

$\Rightarrow$ integral diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{2}{5n-1}$ diverges by the Integral Test

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$ converges or diverges.

$a_n = \frac{1}{n^2+n^3} \Rightarrow f(x) = \frac{1}{x^2+x^3}$

- f is continuous since $x \neq 0, x \neq -1$.
- $f > 0$ on $(1, \infty)$
- f is decreasing since the numerator is constant and den. increases as x increases

$$\int_1^{\infty} \frac{1}{x^2+x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+x^3}\,dx$$

$$\left[\frac{1}{x^2+x^3} = \frac{1}{x^2(1+x)} = \frac{A}{1+x} + \frac{B}{x} + \frac{C}{x^2}\right] x^2(1+x)$$

$$1 = Ax^2 + Bx(1+x) + C(1+x)$$

$x=0$: $1 = 0+0+C \Rightarrow C=1$

$x=-1$: $1 = A + B\cdot 0 + C\cdot 0 \Rightarrow A=1$

$x=1$: $1 = 1\cdot 1^2 + B\cdot 1(2) + 1\cdot(2)$

$1 = 1 + 2B + 2$

$-2 = 2B \Rightarrow B=-1$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{1+x} - \frac{1}{x} + \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\ln|1+x| - \ln|x| - \frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\underbrace{\ln|1+t|}_{\to\infty} - \underbrace{\ln|t|}_{\to\infty} - \cancel{\frac{1}{t}}^{\,0} - \ln 2 + \underbrace{\ln 1}_{=0} + 1\right]$$

$$= \lim_{t\to\infty} \left[\ln\left|\frac{1+t}{t}\right| - \frac{1}{t} - \ln 2 + 1\right] = \lim_{t\to\infty} \left[\ln\left|\frac{1}{t}+1\right| - \frac{1}{t} - \ln 2 + 1\right]$$

$= \ln 1 - 0 - \ln 2 + 1 = -\ln 2 + 1 \Rightarrow$ integral converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$ converges by the Integral Test.

Example. For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? p-series

$a_n = \frac{1}{n^p}$, $f(x) = \frac{1}{x^p}$

- f is continuous since $x \neq 0$
- $f > 0$ on $(1, \infty)$
- f is decreasing since $f'(x) = -px^{-p-1} < 0$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{-1}{1-p}$$

(7.6; $p \neq 1$; $p > 1$ (done in 7.6))

Summary:

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example. The the following series for convergence or divergence:

1. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series, $p = 3 > 1$ $\Rightarrow$ convergent

2. $\sum_{n=1}^{\infty} \frac{5}{n^{\pi}} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ p-series, $p = \pi > 1$ $\Rightarrow$ convergent

3. $\sum_{n=1}^{\infty} n^{-0.7} = \sum_{n=1}^{\infty} \frac{1}{n^{0.7}}$ p-series $p = 0.7 < 1$ divergent

$\sum_{n=1}^{\infty} ar^{n-1}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Note: Be careful not to confuse geometric series (where n or $n-1$ is the power) with p-series where n is the base.

Example. Explain why the Integral Test cannot be used to determine whether the series $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$ is convergent.

- $\cos(\pi n) = -1$ for odd values of n
- $\frac{\cos \pi n}{\sqrt{n}}$ is not decreasing since $\cos \pi n$ alternates between -1 and 1.

Note: $\cos \pi n = (-1)^n$ if n is an integer

Homework 8.3 Solutions: The Integral Test

Prof. Leyda Almodóvar

Name: _____

1. Use the Integral Test to determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty} n^{-0.3}$

   $\sum_{n=1}^{\infty} n^{-0.3} = \sum_{n=1}^{\infty} \frac{1}{n^{0.3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{10}}}$

   Let $f(x) = x^{-0.3} = \frac{1}{x^{\frac{3}{10}}}$. Note that $f(x)$ is continuous because it is a power function and $f(x)$ is clearly positive for $x \geq 1$. Then, $f'(x) = -\frac{3}{10}x^{-\frac{13}{10}} < 0$. Hence, $f(x)$ is also decreasing and we may apply the Integral Test:

   $$\int_1^{\infty} f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t x^{-\frac{3}{10}}\,\mathrm{d}x = \lim_{t\to\infty} \left[\frac{10}{7}x^{\frac{7}{10}}\right]_1^t = \lim_{t\to\infty} \left(\frac{10}{7}t^{\frac{7}{10}} - \frac{10}{7}\right) = \infty$$

   Thus, the series $\sum_{n=1}^{\infty} n^{-0.3}$ diverges by the Integral Test.

   (b) $\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$

   $f(x) = \frac{1}{(3x-1)^4}$ is continuous (rational function), positive, and decreasing (since the denominator increases as n increases) on $[1, \infty)$, so the Integral Test applies.

   $$\int_1^{\infty} \frac{1}{(3x-1)^4}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{1}{(3x-1)^4}\,\mathrm{d}x = \lim_{t\to\infty} \left[-\frac{1}{9}(3x-1)^{-3}\right]_1^t$$
   $$= \lim_{t\to\infty} \left[-\frac{1}{9}(3t-1)^{-3} + \frac{1}{9}[3(1)-1]^{-3}\right] = \lim_{t\to\infty} \left[-\frac{1}{9}(3t-1)^{-3} + \frac{1}{9}\cdot\frac{1}{8}\right] = 0 + \frac{1}{72} = \frac{1}{72}$$

   Note that a simple u-substitution ($u = 3x - 1$) was used to find the integral. Since the improper integral is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$ is also convergent by the Integral Test.

(c) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

$f(x) = x^2 e^{-x^3} = \frac{x^2}{e^{x^3}}$ is continuous, positive and decreasing since

$f'(x) = 2xe^{-x^3} + x^2(-3x^2)e^{-x^3} = xe^{-x^3}(2 - 3x^3) = \frac{x(2-3x^3)}{e^{x^3}} < 0$ for $x > 1$. Hence, the Integral Test applies.

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \Big[-\tfrac{1}{3} e^{-x^3} \Big]_1^t$$

$$= \lim_{t\to\infty} \Big(-\tfrac{1}{3} e^{-t^3} + \tfrac{1}{3} e^{-1} \Big) = \tfrac{1}{3e}$$

Note that u-substitution was used to find the integral ($u = -x^3$). Since the improper integral is convergent, the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is also convergent by the Integral Test.

2. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{\frac{3}{2}}}$

$f(x) = \frac{\sqrt{x}}{1+x^{\frac{3}{2}}}$ is continuous and positive on $[1, \infty)$. Also $f(x)$ is decreasing since

$f'(x) = \frac{1-2x^{\frac{3}{2}}}{2\sqrt{x}(1+x^{\frac{3}{2}})^2} < 0$ for $x \geq 1$ [Derivative found using the quotient rule]

We apply the Integral Test:

$$\int_1^{\infty} \frac{\sqrt{x}}{1+x^{\frac{3}{2}}}\,dx = \lim_{t\to\infty} \int_1^t \frac{\sqrt{x}}{1+x^{\frac{3}{2}}} = \lim_{t\to\infty} \tfrac{2}{3} \int_2^{1+t^{\frac{3}{2}}} \tfrac{1}{u}\,du = \lim_{t\to\infty} \tfrac{2}{3} \Big[\ln|u| \Big]_2^{1+t^{\frac{3}{2}}}$$

$$= \tfrac{2}{3} \lim_{t\to\infty} (\ln|1 + t^{\frac{3}{2}}| - \ln 2) = \infty$$

Hence, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{\frac{3}{2}}}$ diverges by the Integral Test.

(b) $\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$

$f(x) = \frac{3x-4}{x^2-2x} = \frac{3x-4}{x(x-2)} = \frac{2}{x} + \frac{1}{x-2}$ by partial fractions. Note that $f(x)$ is continuous, positive, and decreasing (since both denominators increase as x increases) on $[3, \infty)$.

We apply the Integral Test:

$$\int_3^\infty \tfrac{3x-4}{x^2-2x}\,\mathrm{d}x = \lim_{t\to\infty}\int_3^t \tfrac{2}{x}+\tfrac{1}{x-2}\,\mathrm{d}x = \lim_{t\to\infty}\left[2\int_3^t \tfrac{1}{x}+\int_3^t \tfrac{1}{x-2}\right]$$

$$= \lim_{t\to\infty}\Big[2\ln|x|+\ln|x-2|\Big]_3^t = \lim_{t\to\infty}(2\ln|t|+\ln|t-2|-2\ln 3-\ln 1)=\infty$$

Hence, the series $\sum_{n=3}^{\infty}\frac{3n-4}{n^2-2n}$ diverges by the Integral Test.

(c) $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

$f(x)=\frac{\ln x}{x^2}$ is positive and continuous on $[2,\infty)$. Also, $f(x)$ is decreasing since $f'(x)=\frac{1}{x}\cdot x^{-2}-2x^{-3}\ln x = x^{-3}(1-2\ln x)=\frac{1-2\ln x}{x^3}<0$ when $1-2\ln x<0$, that is, $x>e^{\frac{1}{2}}$

We apply the Integral Test:

$$\int_2^\infty \tfrac{\ln x}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_2^t \tfrac{\ln x}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\left(\Big[-\tfrac{\ln x}{x}\Big]_2^t+\int_2^t \tfrac{1}{x^2}\,\mathrm{d}x\right)$$

$$= \lim_{t\to\infty}\left(-\tfrac{\ln t}{t}+\tfrac{\ln 2}{2}-\Big[\tfrac{1}{x}\Big]_2^t\right) = \lim_{t\to\infty}\left(-\tfrac{\ln t}{t}+\tfrac{\ln 2}{2}-\tfrac{1}{t}+\tfrac{1}{2}\right)=\tfrac{\ln 2+1}{2}$$

Note that we used integration by parts where $u=\ln x$, $\mathrm{d}u=\frac{1}{x}\,\mathrm{d}x$, $\mathrm{d}v=x^{-2}\,\mathrm{d}x$, and $v=-\frac{1}{x}$. Also, we found $\lim_{t\to\infty}-\frac{\ln t}{t}=0$ using L'Hospital's rule.

Hence, the series $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test.

(d) $\sum_{k=1}^{\infty} ke^{-k^2}$

$f(x)=xe^{-x^2}$ is continuous and positive on $[1,\infty)$. Also, $f(x)$ is decreasing since $f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}(1-2x^2)=\frac{1-2x^2}{e^{x^2}}<0$ for $x>\sqrt{\frac{1}{2}}$

We apply the Integral Test:

$$\int_1^\infty xe^{-x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,\mathrm{d}x = \lim_{t\to\infty}\Big[-\tfrac{1}{2}e^{-x^2}\Big]_1^t$$

$$= \lim_{t\to\infty}\left(-\tfrac{1}{2e^{t^2}}+\tfrac{1}{2}e^{-1}\right)=0+\tfrac{1}{2e}=\tfrac{1}{2e}$$

Hence, the series $\sum_{k=1}^{\infty} ke^{-k^2}$ converges by the Integral Test.

3. Explain why the Integral Test can't be used to determine whether the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$ is convergent.

   $f(x) = \frac{\cos^2 x}{1+x^2}$ is not decreasing on $[1, \infty)$. Hence, the hypotheses of the Integral Test are not satisfied for the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$.

8.4 The Comparison Tests

→ usually a geometric series or a p-series

In this section we compare a given series with a series that is known to be convergent or divergent.

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

} Only works if larger series is convergent or smaller series is divergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{7}{3n^2+5n+1}$ converges or diverges.

$a_n = \frac{7}{3n^2+5n+1}$ compare to $\sum_{n=1}^{\infty} \frac{7}{3n^2} = \frac{7}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2>1$ convergent

$\frac{7}{3n^2+5n+1} < \frac{7}{3n^2}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{7}{3n^2+5n+1}$ converges by the Comparison Test

Note: The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. Otherwise, the Comparison Test doesn't apply.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges or diverges.

Compare to $\sum_{n=1}^{\infty} \frac{1}{2^n}$ ← geom. series $r = 1/2$ ⇒ since $|\frac{1}{2}| < 1$

⇒ convergent

$\frac{1}{2^n+1} < \frac{1}{2^n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges by Comparison Test.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges or diverges.

Compare to $\sum_{n=1}^{\infty} \frac{1}{2^n}$ ← convergent geom. series

$\frac{1}{2^n - 1} > \frac{1}{2^n}$ ← Cannot use Comparison Test

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.

Note $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{c} > 0$ so order of division doesn't matter

$a_n = \frac{1}{2^n}$, $b_n = \frac{1}{2^n - 1}$

$\lim_{n\to\infty} \frac{1}{2^n} \div \frac{1}{2^n - 1} = \lim_{n\to\infty} \frac{1}{2^n} \cdot \frac{2^n - 1}{1} = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 > 0$

$\Rightarrow$ Both series are convergent by Limit Comparison Test

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{3n^2 + 7n}{\sqrt{1 + n^5}}$ converges or diverges.

Compare to $\sum_{n=1}^{\infty} \frac{3n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{3n^2}{n^{5/2}} = 3 \cdot \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ p-series $p = 1/2 < 1$ diverges

$\lim_{n\to\infty} \frac{3n^2 + 7n}{\sqrt{1+n^5}} \cdot \frac{n^{1/2}}{3} = \lim_{n\to\infty} \frac{3n^{5/2} + 7n^{3/2}}{3\sqrt{1+n^5}} = \lim_{n\to\infty} \frac{\frac{3n^{5/2}}{n^{5/2}} + \frac{7n^{3/2}}{n^{5/2}}}{3\sqrt{\frac{1}{n^5} + 1}}$

$= \lim_{n\to\infty} \frac{3 + \frac{7}{n}}{3\sqrt{\frac{1}{n^5} + 1}} = \frac{3}{3} = 1 > 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{3n^2 + 7n}{\sqrt{1+n^5}}$ diverges by the LCT

Example. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges or diverges.

Compare to $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2}} = \sum_{n=2}^{\infty} \frac{1}{n^2}$ p-series $p=2>1$ convergent

$\frac{1}{n\sqrt{n^2-1}} > \frac{1}{n\sqrt{n^2}}$ Direct comparison doesn't work!

$\lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n\sqrt{n^2-1}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n} \overset{\downarrow}{=} \lim_{n\to\infty} \frac{\frac{\sqrt{n^2-1}}{\sqrt{n^2}}}{\frac{n}{n}}$ $\quad n=\sqrt{n^2}$

$= \lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n^2}}}{1} \to 0 = \sqrt{1} = 1 > 0$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by the LCT

Homework 8.4 Solutions: The Comparison Tests

Prof. Leyda Almodóvar

Name: _____

1. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be divergent.

   (a) If $a_n > b_n$ for all n, what can you say about $\sum a_n$? Why?

   $\sum a_n$ is divergent by the Comparison Test.

   (b) If $a_n < b_n$ for all n, what can you say about $\sum a_n$? Why?

   We cannot say anything about $\sum a_n$ because the Comparison Test doesn't apply.

2. Determine whether the series is converges or diverges.

   (a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

   Note that $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ for all $n \geq 2$, and $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series since $p = \frac{1}{2} < 1$. Hence, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges by the Comparison Test.

   (b) $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

   Note that $\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series since $p = 2 > 1$. Hence, $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ converges by the Comparison Test.

   (c) $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

Note that $\frac{6^n}{5^n-1} > \frac{6^n}{5^n}$ and $\sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a divergent geometric series since $r = \frac{6}{5} > 1$. Hence, $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ diverges by the Comparison Test.

(d) $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$

Note that $\frac{k \sin^2 k}{1+k^3} \leq \frac{k}{1+k^3} < \frac{k}{k^3} = \frac{1}{k^2}$ for all $k \geq 1$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series since $p = 2 > 1$. Hence, $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$ converges by the Comparison Test.

(e) $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

Note that $\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \frac{2k(k^2)}{k(k^2)^2} = \frac{2k^3}{k^5} = \frac{2}{k^2}$ for all $k \geq 1$, and $\sum_{k=1}^{\infty} \frac{2}{k^2}$ is a convergent p-series since $p = 2 > 1$. Hence, $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges by the Comparison Test.

(f) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$

Note that $\frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3n^4}} < \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{\frac{4}{3}}}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ is a convergent p-series since $p = \frac{4}{3} > 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ converges by the Comparison Test.

(g) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Note that $\frac{1}{n^n} \leq \frac{1}{n^2}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series since $p = 2 > 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the Comparison Test.

(h) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

We apply the Limit Comparison Test with $a_n = \frac{n^2+n+1}{n^4+n^2}$ and $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n+1}{n^4+n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^2+n+1}{n^2+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series since $p = 2 > 1$, $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ is convergent as well by the Limit Comparison Test.

(i) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$

Note that $\frac{e^{\frac{1}{n}}}{n} > \frac{1}{n}$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series since $p = 1$. Hence, $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ diverges by the Comparison Test.

(j) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Note that $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \ldots (n-1)n}{n \cdot n \cdot n \ldots n} \leq \frac{1}{n} \cdot \frac{2}{n} = \frac{2}{n^2}$ for $n \geq 2$. Since, $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is a convergent p-series ($p = 2 > 1$), $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent as well by the Comparison Test.

8.5 The Alternating Series Test

An **alternating series** is a series whose term are alternately positive and negative.

Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 + \ldots$ satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

→ or $(-1)^n$ or $(-1)^{n+1}$

* If (i) or (ii) fail, the series diverges by the Test for Divergence.

Example. Test the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ for convergence or divergence.

$a_n = (-1)^{n-1}\cdot\frac{1}{n} \Rightarrow b_n = \frac{1}{n}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$

Also, $\frac{1}{n+1} < \frac{1}{n} \Rightarrow b_n$ is decreasing

$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test.

Example. Test the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{5n+1}$ for convergence or divergence.

$b_n = \frac{4n}{5n+1}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{4n}{5n+1} \overset{H}{=} \lim_{n\to\infty}\frac{4}{5+0} = \frac{4}{5} \neq 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(-1)^{n-1}4n}{5n+1} = \lim_{n\to\infty}(-1)^{n-1}\cdot\frac{4n}{5n+1} = \text{D.N.E.}$

$(-1)^{n-1}$: oscillating between -1 and 1; $\frac{4n}{5n+1} \to 4/5$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{5n+1}$ diverges by the Test for Divergence.

Example. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$ for convergence or divergence.

$b_n = \frac{n^2}{n^3+1}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = \frac{0}{1} = 0$

$f(x) = \frac{x^2}{x^3+1} \to f'(x) = \frac{(x^2)'(x^3+1) - (x^3+1)'x^2}{(x^3+1)^2} = \frac{2x(x^3+1) - 3x^2 x^2}{(x^3+1)^2}$

$= \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2} < 0 \Rightarrow -x^4+2x < 0$

$x(2-x^3) < 0$

$x > 0$ and $2-x^3 < 0$ or $x < 0$ and $2-x^3 > 0$

$2 < x^3$

$\sqrt[3]{2} < x$

$\Rightarrow f$ is ultimately decreasing

$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ converges by the AST

Example. Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$ for convergence or divergence.

$b_n = ne^{-n}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{e^n} = 0$

$(b_n)' = (n)'e^{-n} + (e^{-n})'n = 1 \cdot e^{-n} - e^{-n} \cdot n = e^{-n}(1-n) < 0$

when $1-n < 0 \Rightarrow n > 1$

$\Rightarrow b_n$ is decreasing

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} ne^{-n}$ converges by the AST

Homework 8.5: Alternating Series

Prof. Leyda Almodóvar

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1. Test the series for convergence or divergence.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

   $b_n = \frac{1}{\sqrt{n+1}}$. Note b_n is decreasing since $\frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}}$ for all n (the denominator is increasing as n increases while the numerator stays constant.)

   $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

   By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ is convergent.

   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$

   $b_n = \frac{\sqrt{n}}{2n+3}$. Since both the numerator and the denominator are increasing as n increases, it is not clear whether b_n is decreasing, increasing, or neither.

   Consider the derivative of $f(x) = \frac{\sqrt{x}}{2x+3}$:

   $$f'(x) = \frac{\frac{1}{2}x^{-\frac{1}{2}}(2x+3) - 2\sqrt{x}}{(2x+3)^2} = \frac{x^{-\frac{1}{2}}(x+\frac{3}{2}-2x)}{(2x+3)^2} = \frac{-x+\frac{3}{2}}{\sqrt{x}(2x+3)^2}$$

   Note that the denominator $\sqrt{x}(2x+3)^2$ is always positive. Hence, $f'(x) < 0$ if $-x + \frac{3}{2} < 0$, that is, $x > \frac{3}{2}$. Thus, $f(x)$ is decreasing, which implies b_n is decreasing. Then,

   $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{2n+3} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{2+\frac{3}{n}} = \frac{0}{2+0} = 0$$

   By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$ is convergent.

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

$b_n = \frac{n}{e^n}$. Since both the numerator and the denominator are increasing as n increases, it is not clear whether b_n is decreasing, increasing, or neither.

Consider the derivative of $f(x) = \frac{x}{e^x}$:

$$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} = \frac{1-x}{e^x}$$

Note that $e^x > 0$. Hence, $f'(x) < 0$ when $1 - x < 0$, that is, $x > 1$. Thus, $f(x)$ is decreasing, which implies b_n is decreasing.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0 \text{ by L'Hospital's rule.}$$

By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$ is convergent.

(d) $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

$b_n = \arctan n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \arctan n \text{ does not exist.}$$

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$ is divergent by the Test for Divergence.

(e) $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

Note that $\cos n\pi = (-1)^n$. Hence, the series may be rewritten as

$$\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

Then, $b_n = \frac{n}{2^n}$. Since both the numerator and the denominator are increasing as n increases, it is not clear whether b_n is decreasing, increasing, or neither.

Consider the derivative of $f(x) = \frac{x}{2^x}$:

$$f'(x) = \frac{2^x - x(\ln 2)2^x}{(2^x)^2} = \frac{2^x(1 - x\ln 2)}{(2^x)^2} = \frac{1 - x\ln 2}{2^x}.$$

Note that $2^x > 0$. Hence, $f'(x) < 0$ when $1 - x \ln 2 < 0$, that is $x > \frac{1}{\ln 2}$. Thus, $f(x)$ is decreasing, which implies b_n is decreasing.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \tfrac{n}{2^n} = \lim_{n\to\infty} \tfrac{1}{2^n} = 0 \text{ by L'Hospital's rule.}$$

By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n}$ is convergent.

(f) $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cos(\tfrac{\pi}{n}) = \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \cos(\tfrac{\pi}{n}) \text{ does not exist since } \lim_{n\to\infty} \cos(\tfrac{\pi}{n}) = 1$$

Hence, $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$ is divergent by the Test for Divergence.

(g) $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$

$b_n = \sqrt{n+1} - \sqrt{n}$. Consider the derivative of $f(x) = \sqrt{x+1} - \sqrt{x}$:

$$f'(x) = \tfrac{1}{2\sqrt{x+1}} - \tfrac{1}{2\sqrt{x}} < 0 \text{ when } \tfrac{1}{2\sqrt{x+1}} < \tfrac{1}{2\sqrt{x}}, \text{ which is always true.}$$

Hence, $f(x)$ is decreasing, which implies b_n is decreasing.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \tfrac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \tfrac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

By the Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$ is convergent.

8.6 Absolute Convergence and the Ratio and Root Tests

Definition. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Definition. A series $\sum a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent.

Example. Determine whether the following series are absolute convergent, conditionally convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2>1$ convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ p-series $p=1$ divergent

We tested $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ in the previous section and found it was convergent. Hence, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

3. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$

We can do a Comparison Test

$|\cos n| \le 1 \Rightarrow \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series $p=2>1$ convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent by the Comparison Test

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

The Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive; that is, no conclusion can be made about the convergence or divergence of $\sum a_n$. *Won't work for p-series or series that resemble p-series.

Example. Test the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ for absolute convergence.

$a_n = \frac{(-1)^n n^3}{3^n}$, $a_{n+1} = \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{(-1)^n n^3}\right| = \lim_{n\to\infty}\left|\frac{-(n+1)^3}{3\cdot n^3}\right|$$

$$= \lim_{n\to\infty}\left|-\frac{1}{3}\left(\frac{n+1}{n}\right)^3\right| = \lim_{n\to\infty}\left|-\frac{1}{3}\cdot\left(1+\frac{1}{n}\right)^3\right| = \frac{1}{3} < 1$$

$\Rightarrow \sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ is absolutely convergent by the Ratio Test.

Example. Test the series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ for convergence or divergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| = \lim_{n\to\infty}\left|\frac{1}{5}\cdot\frac{n+1}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{5}\cdot\left(1+\frac{1}{n}\right)\right| = \frac{1}{5} < 1$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{n}{5^n}$ converges by the Ratio Test.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n^4 4^n}{n!}$ for convergence or divergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4\cdot 4^{n+1}}{(n+1)!}\cdot\frac{n!}{n^4\cdot 4^n}\right| = \lim_{n\to\infty}\left|4\cdot\left(\frac{n+1}{n}\right)^4\cdot\frac{n!}{(n+1)!}\right|$$

$(n+1)! = (n+1)\cdot n(n-1)\cdots 3\cdot 2\cdot 1 = (n+1)\cdot n!$

$$= \lim_{n\to\infty}\left|4\cdot\left(1+\frac{1}{n}\right)^4\cdot\frac{n!}{(n+1)\cdot n!}\right| = \lim_{n\to\infty}\left|4\cdot\left(1+\frac{1}{n}\right)^4\cdot\frac{1}{n+1}\right| = 0 < 1$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{n^4 4^n}{n!}$ converges by the Ratio Test

Example. Test the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ for convergence or divergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{e^{n+1}}\cdot\frac{e^n}{n} = \lim_{n\to\infty}\frac{1}{e}\cdot\frac{n+1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{e}\cdot\left(1+\frac{1}{n}\right)^{\to 0} = \frac{1}{e} < 1$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{n}{e^n}$ converges by the Ratio Test.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ for convergence or divergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!} = \lim_{n\to\infty}\frac{(n+1)\cdot n!}{n!}\cdot\frac{1}{100}$$

$$= \lim_{n\to\infty}\frac{(n+1)}{100} = \infty$$

$\Rightarrow \sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges by the Ratio Test.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ for convergence or divergence.

HW Problem: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

convergent p-series $p=2>1$

$\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdots n\cdots n\cdot n} = \frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{3}{n}\cdot\frac{2}{n}\cdot\frac{1}{n} \le \frac{2}{n^2}$ ($\frac{n-1}{n}<1$, $\frac{n-2}{n}<1$, $\frac{3}{n}<1$, $\frac{2}{n}<1$, $\frac{1}{n}<1$)

$\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}$

$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\cdot n!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\frac{\ln(1+\frac{1}{n})}{1/n}} \overset{H}{=} \lim_{n\to\infty} e^{\frac{\frac{1}{1+1/n}\cdot\left(-\frac{1}{n^2}\right)}{\left(-1/n^2\right)}}$

1^∞ ind. power

$y = \left(1+\frac{1}{n}\right)^n \Rightarrow y = e^{\frac{n\cdot\ln(1+\frac{1}{n})}{1/n}}$

$\ln y = \ln\left(1+\frac{1}{n}\right)^n = e^{\frac{\ln(1+\frac{1}{n})}{1/n}}$

$\ln y = n\cdot\ln\left(1+\frac{1}{n}\right)$

$= \lim_{n\to\infty} e^{\frac{1}{1+1/n}} = e > 1$

$\Rightarrow$ diverges by the Ratio Test.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive; that is, no conclusion can be made about the convergence or divergence of $\sum a_n$.

Example. Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$.

*Not a geom. series ("r" is not constant)

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|}=\lim_{n\to\infty}\frac{2n+3}{3n+2}$$

$$\stackrel{\text{H}}{=}\lim_{n\to\infty}\frac{2}{3}=\frac{2}{3}<1 \Rightarrow \text{convergent by the Root Test}$$

Example. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{5^{2n}}{10^{n+1}}$.

$10^{n+1}=10^n\cdot 10^1$

$$a_n=\frac{5^{2n}}{10^{n+1}}=\frac{(5^2)^n}{10\cdot 10^n}=\frac{1}{10}\cdot\left(\frac{25}{10}\right)^n$$

$$\sum_{n=1}^{\infty}\frac{5^{2n}}{10^{n+1}}=\frac{1}{10}\cdot\sum_{n=1}^{\infty}\left(\frac{25}{10}\right)^n=\frac{1}{10}\cdot\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^n$$

Option #1: geom. series $r=\frac{5}{2}>1 \Rightarrow$ divergent

Option #2: Root Test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{25}{10}\right|^n}=\frac{25}{10}=\frac{5}{2}>1 \Rightarrow \text{divergent}$$

Homework 8.6: Absolute Convergence and the Ratio and Root Tests

Prof. Leyda Almodóvar

Name: _____

1. Determine whether the series is absolutely convergent or conditionally convergent.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

   $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$, then $|a_n| = |\frac{(-1)^{n-1}}{\sqrt{n}}| = \frac{1}{\sqrt{n}}$

   $$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ is a divergent } p\text{-series } (p = \tfrac{1}{2} < 1)$$

   We test the original series using the Alternating Series Test:

   $b_n = \frac{1}{\sqrt{n}}$ is clearly decreasing since the denominator is increasing as n increases while the numerator stays constant.

   $$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

   Hence, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent by the Alternating Series Test.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^3+1}$

   $a_n = \frac{(-1)^{n}}{n^3+1}$ and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^3+1}$

   Note that $|a_n| = \frac{1}{n^3+1} < \frac{1}{n^3}$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges by the Comparison Test since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series $(p = 3 > 1)$

Hence, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ is absolutely convergent.

(c) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

$a_n = (-1)^{n-1} \frac{n}{n^2+4}$ and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$

Note $\frac{n}{n^2+4} < \frac{n}{n^2} = \frac{1}{n}$ but the Comparison Test doesn't work since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. We apply the Limit Comparison Test:

$$\lim_{n\to\infty} \left| \frac{1}{n} \cdot \frac{n^2+4}{n} \right| = \lim_{n\to\infty} \left| \frac{n^2+4}{n^2} \right| = \lim_{n\to\infty} \left| 1 + \frac{4}{n^2} \right| = 1 > 0$$

Hence, $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges by the Limit Comparison Test.

Alternatively, we could apply the Integral Test:

$f(x) = \frac{x}{x^2+4}$ is continuous and positive on $[1, \infty)$. $f(x)$ is also decreasing:

$$f'(x) = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2}$$

Note that the denominator is always positive. Hence, $f'(x) < 0$ when $-x^2+4 < 0$, that is, when $x < -2$ (interval not included in the series) and when $x > 2$. Then,

$$\int_1^{\infty} \frac{x}{x^2+4}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+4}\,\mathrm{d}x = \lim_{t\to\infty} \left[\frac{1}{2} \ln|x^2+4| \right]_1^t = \lim_{t\to\infty} \left[\frac{1}{2} \ln|t^2+4| - \frac{1}{2} \ln 5 \right] = \infty$$

Note that u-substitution was used with $u = x^2 + 4$.

Hence, $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges by the Integral Test. Now we apply the Alternating Series Test to the original series:

$b_n = \frac{n}{n^2+4}$ is decreasing (shown above) and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+4} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{4}{n^2}} = \frac{0}{1} = 0$$

Thus, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ is conditionally convergent by the Alternating Series Test.

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

We apply the Ratio Test:

$$\lim_{n\to\infty} |\tfrac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\tfrac{2^{n+1}}{(n+1)^2} \cdot \tfrac{n^2}{2^n}| = \lim_{n\to\infty} |\tfrac{2n^2}{(n+1)^2}| = 2 \lim_{n\to\infty} \tfrac{1}{(1+\frac{1}{n})^2} = 2 > 1$$

Hence, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ is divergent by the Ratio Test.

(e) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

We apply the Ratio Test:

$$\lim_{n\to\infty} |\tfrac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\tfrac{3^{n+1}}{(2(n+1)+1)!} \cdot \tfrac{(2n+1)!}{3^n}| = \lim_{n\to\infty} |\tfrac{3(2n+1)!}{(2n+3)(2n+2)(2n+1)!}| = \lim_{n\to\infty} |\tfrac{3}{(2n+3)(2n+2)}| = 0 < 1$$

Hence, $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ is absolutely convergent by the Ratio Test.

(f) $\sum_{k=1}^{\infty} ke^{-k}$

We apply the Ratio Test. Note that $\frac{k}{e^k} > 0$ so we don't need absolute values.

$$\lim_{k\to\infty} |\tfrac{a_{k+1}}{a_k}| = \lim_{k\to\infty} \tfrac{k+1}{e^{k+1}} \cdot \tfrac{e^k}{k} = \lim_{k\to\infty} \tfrac{k+1}{k} \cdot \tfrac{e^k}{e^{k+1}} = \lim_{k\to\infty} (1 + \tfrac{1}{k}) \cdot \tfrac{1}{e} = \tfrac{1}{e} < 1$$

Hence, $\sum_{k=1}^{\infty} ke^{-k}$ is absolutely convergent by the Ratio Test.

(g) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

We apply the Ratio Test. Note that $\frac{n!}{100^n} > 0$ so we don't need absolute values.

$$\lim_{n\to\infty} |\tfrac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \tfrac{(n+1)!}{100^{n+1}} \cdot \tfrac{100^n}{n!} = \lim_{n\to\infty} \tfrac{(n+1)n!}{100 \cdot n!} = \lim_{n\to\infty} \tfrac{n+1}{100} = \infty$$

Hence, $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ diverges by the Ratio Test.

(h) $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$

We apply the Ratio Test.

$$\lim_{n\to\infty} |\tfrac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\tfrac{(n+1)^{10}}{10^{n+2}} \cdot \tfrac{10^{n+1}}{n^{10}}| = \lim_{n\to\infty} |\tfrac{(n+1)^{10}}{10n^{10}}| = \lim_{n\to\infty} |\tfrac{(1+\frac{1}{n})^{10}}{10}| = \tfrac{1}{10} < 1$$

Hence, $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$ is absolutely convergent by the Ratio Test.

(i) $\sum_{n=1}^{\infty} \frac{n!}{n}$

We apply the Ratio Test. Note that $\frac{n!}{n} > 0$ so we don't need absolute values.

$$\lim_{n\to\infty} |\tfrac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \tfrac{(n+1)!}{n+1} \cdot \tfrac{n}{n!} = \lim_{n\to\infty} \tfrac{(n+1)n!}{(n+1)} \cdot \tfrac{n}{n!} = \lim_{n\to\infty} n = \infty$$

Hence, $\sum_{n=1}^{\infty} \frac{n!}{n}$ is divergent by the Ratio Test.

2. Use the Root Test to determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

Note $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} (\frac{-2}{n})^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \tfrac{2}{n} = 0 < 1$$

Hence, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

(b) $\sum_{n=1}^{\infty} (\frac{-2n}{n+1})^{5n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\tfrac{2n}{n+1}\right)^5 = \lim_{n\to\infty} \left(\tfrac{2}{1+\frac{1}{n}}\right)^5 = \left(\tfrac{2}{1+0}\right)^5 = 32 > 1$$

Hence, $\sum_{n=1}^{\infty} (\frac{-2n}{n+1})^{5n}$ diverges by the Root Test.

8.7 Strategy for Testing Series

Classify the series according to its form.

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.
   Examples: $\sum \frac{1}{n^3}$, $\sum n^{-\frac{1}{2}}$, $\sum \frac{3}{n^\pi}$
2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
   Examples: $\sum \left(\frac{2}{3}\right)^n$, $\sum 4^{2n}3^{n-1}$, $\sum 2^{1-n}$
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered.
   - If a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. The value of p should be chosen by keeping only the highest powers of n in the numerator and denominator.
     Examples: $\sum \frac{\sqrt{7n^2+3}}{\sqrt{5n^3-1}}$, $\sum \frac{5n^4-3n+2}{2n^5-1}$
   - The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$ then the Alternating Series Test is appropriate.
6. Series that involve factorials or other products (including a constant raised to the n-th power) are often conveniently tested using the Ratio Test.
   Examples: $\sum \frac{3^n}{n!}$, $\sum \frac{(n+1)!}{n^2}$
   Note: Recall that the Ratio Test is inconclusive if $\left|\frac{a_{n+1}}{a_n}\right| = 1$ as $n \to \infty$. Do not use for p-series or algebraic functions of n.
7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
   Examples: $\sum \left(\frac{5n+7}{4n-1}\right)^n$, $\sum \left(\frac{n+1}{3n-2}\right)^{5n}$
8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated and f is continuous, positive, and decreasing, then the Integral Test is effective.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ for convergence or divergence.

Test for Divergence

$\lim_{n\to\infty} \frac{n+1}{3n+1} \neq \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$

$\Rightarrow$ Divergent

Example. Test the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ for convergence or divergence.

Compare to $\sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{3n^3} = \sum_{n=1}^{\infty} \frac{n^{3/2}}{3n^3} = \frac{1}{3} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ p-series $p = 3/2 > 1$ $\Rightarrow$ convergent

L.C.T.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{\frac{n^{3/2}\sqrt{n^3+1}}{n^3}}{\frac{3n^3}{n^3}+\frac{4n^2}{n^3}+\frac{2}{n^3}}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{n^{3/2}}}{3+4/n+2/n^3} = \lim_{n\to\infty} \frac{\sqrt{1+1/n^3}}{3+4/n+\frac{2}{n^3}} = \frac{1}{3} > 0$$

$\Rightarrow$ By the LCT, $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ also converges.

Example. Test the series $\sum_{n=1}^{\infty} ne^{-n^2}$ for convergence or divergence.

Integral Test or Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{e^{n^2}}{e^{n^2+2n+1}}\right|$$

$$= \lim_{n\to\infty} \left|(1+\tfrac{1}{n}) \cdot \frac{1}{e^{2n+1}}\right| = 0 < 1 \Rightarrow \text{convergent by Ratio Test}$$

Example. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^4}{n^5+2}$ for convergence or divergence.

converges by AST

AST

$b_n = \frac{n^4}{n^5+2}$ $\lim_{n\to\infty} \frac{n^4}{n^5+2} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{2}{n^5}} = \frac{0}{1} = 0$

$(b_n)' = \frac{(n^4)'(n^5+2) - (n^5+2)' n^4}{(n^5+2)^2} = \frac{4n^3(n^5+2) - 5n^4 \cdot n^4}{(n^5+2)^2}$

$= \frac{-n^8+8n^3}{(n^5+2)^2} < 0$ when $-n^8+8n^3 < 0$

$n^3(-n^5+8) < 0$

$\Rightarrow n^3 > 0$ and $-n^5+8 < 0$

$\Rightarrow n > 0$ $8 > n^5$ $\sqrt[5]{8} > n$ $\Rightarrow$ b_n is ultimately decreasing

Example. Test the series $\sum_{n=1}^{\infty} \frac{1}{4+5^n}$ for convergence or divergence.

Compare to $\sum_{n=1}^{\infty} \frac{1}{5^n}$ geom. series $r = 1/5 < 1 \Rightarrow$ convergent

$\frac{1}{4+5^n} < \frac{1}{5^n}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4+5^n}$ converges by Comparison Test

Example. Test the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ for convergence or divergence.

$\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)=\sum_{n=1}^{\infty}\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{1}{3^n}$

$\sum_{n=1}^{\infty}\frac{1}{n^3}$: p-series, $p=3>1$ $\Rightarrow$ convergent

$\sum_{n=1}^{\infty}\frac{1}{3^n}$: geom. series, $-1<r=1/3<1$ $\Rightarrow$ convergent

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ converges

Example. Test the series $\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$ for convergence or divergence.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2}\right|=\lim_{n\to\infty}\left|\frac{3\cdot(n+1)^2\cdot n!}{(n+1)\cdot n!\ n^2}\right|$$

$$=\lim_{n\to\infty}\left|\frac{3\cdot(n+1)}{n^2}\right|=\lim_{n\to\infty}\left|3\cdot\left(\frac{1}{n}+\frac{1}{n^2}\right)\right|=0<1$$

$\Rightarrow$ convergent

Example. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^4}{4^n}$ for convergence or divergence.

Ratio Test or AST

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1-1}(n+1)^4}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n-1} n^4} \right| = \lim_{n\to\infty} \left| -\left(\frac{n+1}{n}\right)^4 \cdot \frac{1}{4} \right|$$

$$= \lim_{n\to\infty} \left| -\left(1+\frac{1}{n}\right)^4 \cdot \frac{1}{4} \right| = \frac{1}{4} < 1 \Rightarrow \text{convergent}$$

Example. Test the series $\sum_{n=1}^{\infty} \frac{2^{n-1} 3^{n+1}}{n^n}$ for convergence or divergence.

*not a geometric series

$$\sum_{n=1}^{\infty} \frac{2^{n-1} 3^{n+1}}{n^n} = \sum_{n=1}^{\infty} \frac{2^n \cdot 2^{-1} \cdot 3^n \cdot 3}{n^n} = 2^{-1} \cdot 3 \sum_{n=1}^{\infty} \underbrace{\frac{2^n \cdot 3^n}{n^n}}_{a_n}$$

Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n \cdot 3^n}{n^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2\cdot 3}{n}\right)^n} = \lim_{n\to\infty} \frac{6}{n} = 0 < 1$$

$\Rightarrow$ convergent

Homework 8.7: Strategy for Testing Series

Prof. Leyda Almodóvar

Name: _____

1. Test the series for convergence or divergence.

   (a) $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

   $a_n = \frac{n-1}{n^3+1} < \frac{n-1}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$

   Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ is also convergent by the Comparison Test.

   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$

   Let $f(x) = \frac{x^2-1}{x^2+1}$. Note that $f'(x) = \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} > 0$ so $f(x)$ is increasing. Hence, the Alternating Series Test doesn't work.

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1} \text{ does not exist.}$$

   Hence, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ diverges by the Test for Divergene.

   (c) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

   $a_n = (-1)^{n-1} \frac{n}{n^2+4}$ and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+4}$

   Note $\frac{n}{n^2+4} < \frac{n}{n^2} = \frac{1}{n}$ but the Comparison Test doesn't work since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. We apply the Integral Test:

   $f(x) = \frac{x}{x^2+4}$ is continuous and positive on $[1, \infty)$. $f(x)$ is also decreasing:

$$f'(x) = \frac{x^2+4-2x^2}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2}$$

Note that the denominator is always positive. Hence, $f'(x) < 0$ when $-x^2+4 < 0$, that is, when $x < -2$ (interval not included in the series) and when $x > 2$. Then,

$$\int_1^\infty \frac{x}{x^2+4}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+4}\,\mathrm{d}x = \lim_{t\to\infty}\left[\frac{1}{2}\ln|x^2+4|\right]_1^t =$$
$$\lim_{t\to\infty}\left[\frac{1}{2}\ln|t^2+4| - \frac{1}{2}\ln 5\right] = \infty$$

Note that u-substitution was used with $u = x^2+4$.

Hence, $\sum\limits_{n=1}^{\infty} \frac{n}{n^2+4}$ diverges by the Integral Test. Now we apply the Alternating Series Test to the original series:

$b_n = \frac{n}{n^2+4}$ is decreasing (shown above) and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+4} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{4}{n^2}} = \frac{0}{1} = 0$$

Thus, $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\frac{n}{n^2+4}$ is conditionally convergent by the Alternating Series Test.

(d) $\sum\limits_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

Note that $\sum\limits_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}} = \sum\limits_{n=1}^{\infty} \left(\frac{n^2}{(1+n)^3}\right)^n$ We apply the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2}{(1+n)^3} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\left(\frac{1}{n}+1\right)^3} = \frac{0}{1} = 0 < 1$$

Hence, $\sum\limits_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ is absolutely convergent by Root Test.

(e) $\sum\limits_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$

$a_k = \frac{1}{k\sqrt{k^2+1}} < \frac{1}{k\sqrt{k^2}} = \frac{1}{k^2}$

$\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2 > 1$)

Hence, $\sum\limits_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ is also convergent by the Comparison Test.

(f) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Note $\sum_{n=1}^{\infty} n^2 e^{-n^3} = \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

We apply the Ratio Test.

$\lim_{n\to\infty} \frac{(n+1)^2}{e^{(n+1)^3}} \cdot \frac{e^{n^3}}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{e^{n^3}}{e^{(n+1)^3}} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{1} \cdot \frac{1}{e^{-3n^2-3n-1}} = 1 \cdot 0 = 0 < 1$

Hence, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is absolutely convergent by the Ratio Test.

(g) $\sum_{n=1}^{\infty} \frac{\sin n}{1+2^n}$

We know $-1 \le \sin n \le 1$. Hence, $|\frac{\sin n}{1+2^n}| \le \frac{1}{1+2^n} < \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$)

Hence, $\sum_{n=1}^{\infty} \frac{|\sin n|}{1+2^n}$ also converges by the Comparison Test.

Thus, $\sum_{n=1}^{\infty} \frac{\sin n}{1+2^n}$ is absolutely convergence and hence convergent.

(h) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$

$a_n = \frac{\sqrt{n^4+1}}{n^3+n}$

We apply the Limit Comparison Test with $b_n = \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n}$

$$\lim_{n\to\infty} |\frac{a_n}{b_n}| = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^3+n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2+1} = \lim_{n\to\infty} \frac{\sqrt{\frac{n^4}{n^4}+\frac{1}{n^4}}}{1+\frac{1}{n^2}} = \frac{1+0}{1+0} = 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p = 1$), $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ is divergent as well by the Limit Comparison Test.

(i) $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$

$a_k = \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)} < \frac{\sqrt[3]{k}}{k(\sqrt{k}+1)} < \frac{\sqrt[3]{k}}{k\sqrt{k}} = \frac{k^{\frac{1}{3}}}{k^{\frac{3}{2}}} = \frac{1}{k^{\frac{7}{6}}}$

$\sum_{k=1}^{\infty} \frac{1}{k^{\frac{7}{6}}}$ is a convergent p-series ($p = \frac{7}{6} > 1$), hence $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$ is convergent by

the Comparison Test.

(j) $\sum_{n=1}^{\infty} n\sin(\frac{1}{n})$

Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}[-\cos(\frac{1}{n})]}{-\frac{1}{n^2}}$$
$$= -\cos(0) = -1 \neq 0$$

Hence, $\sum_{n=1}^{\infty} n\sin(\frac{1}{n})$ diverges by the Test for Divergence.

(k) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$

Let $f(x) = \frac{e^{\frac{1}{x}}}{x^2}$. Note that $f(x)$ is positive and continuous on $[1,\infty)$. Also, $f(x)$ is decreasing since:

$$f'(x) = \frac{e^{\frac{1}{x}}(-\frac{1}{x^2})x^2 - 2xe^{\frac{1}{x}}}{x^4} = \frac{-e^{\frac{1}{x}} - 2xe^{\frac{1}{x}}}{x^4} = \frac{-e^{\frac{1}{x}}(1+2x)}{x^4} < 0 \text{ for } x > -\frac{1}{2}$$

We apply the Integral Test:

$$\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty}\left[-e^{\frac{1}{x}}\right]_1^t = \lim_{t\to\infty}\left(-e^{\frac{1}{t}} + e^1\right) = -e^0 + e = e - 1$$

Note that u-substitution was used with $u = \frac{1}{x}$.

Since the improper integral $\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx$ converges, the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ converges as well by the Integral Test.

8.8 Power Series

Definition. A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, where x is a variable and the c_n's are constants is called a **power series centered at** a.

- For each fixed x, the series is a series of constants that we can test for convergence or divergence.
- A power series may converge for some values of x and diverge for other values of x.
- For example, if we let $a = 0$ and $c_n = 1$ for all n, then the series becomes the geometric series $\sum_{n=0}^{\infty}(x)^n = 1 + x + x^2 + x^3 + \ldots$ which converges when $|x| < 1$ and diverges when $|x| \geq 1$.
- We test these series using the Ratio Test or Root Test when appropriate.

Example. For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

We apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!\,x}{n!}\right| = \lim_{n\to\infty}|(n+1)\cdot x|$$

$$= |x|\cdot\lim_{n\to\infty}|n+1| = \infty$$

The limit does not depend on x

$\Rightarrow \sum_{n=0}^{\infty} n!x^n$ diverges for all $x \neq 0$.

$R = 0$

$I = \{0\}$

Example. For what values of x is the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ convergent?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-x^{2n+2}\cdot 2^{2n}(n!)^2}{2^{2n+2}[(n+1)!]^2 x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{-x^2\cdot(n!)^2}{2^2[(n+1)\cdot n!]^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-x^2\cdot(n!)^2}{4\cdot(n+1)^2\cdot(n!)^2}\right| = \lim_{n\to\infty}\left|\frac{-x^2}{4(n+1)^2}\right| = \left|\frac{-x^2}{4}\right|\cdot\lim_{n\to\infty}\left|\frac{1}{(n+1)^2}\right| = 0 < 1$$

$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ converges for all x by the Ratio Test

$R = \infty$

$I = (-\infty, \infty)$

Example. For what values of x is the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$ convergent?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1}\cdot\frac{n}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)\cdot n}{n+1}\right|$$

$$= |x-2|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| \overset{\#}{=} |x-2|\cdot\lim_{n\to\infty}\frac{1}{1+0} = |x-2| \qquad |x-a|<R$$

By the Ratio Test, the series converges when $|x-2|<1 \Rightarrow R=1$

$$\Rightarrow -1<x-2<1$$

$$1<x<3$$

* The Ratio Test is inconclusive when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

$\Rightarrow$ We need to test $x=1$ and $x=3$ separately.

$x=1$: $\sum_{n=0}^{\infty}\frac{(1-2)^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ AST: $b_n=\frac{1}{n}$

$\frac{1}{n+1}<\frac{1}{n} \Rightarrow b_n$ is decreasing

$\lim_{n\to\infty}\frac{1}{n}=0 \Rightarrow$ convergent

$x=3$: $\sum_{n=0}^{\infty}\frac{(3-2)^n}{n} = \sum_{n=0}^{\infty}\frac{1^n}{n} = \sum_{n=0}^{\infty}\frac{1}{n}$ $\rightarrow$ p-series, $p=1$ $\Rightarrow$ divergent } or harmonic series

$\Rightarrow \sum_{n=0}^{\infty}\frac{(x-2)^n}{n}$ converges when $1\le x<3$.

$\Rightarrow I=[1,3)$

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

Definition. The **interval of convergence** I of a power series is the interval that consists of all values of x for which the series converges.

(i) $I = \{a\}$

(ii) $I = (-\infty, \infty)$

(iii) $|x-a| < R \Rightarrow -R < x-a < R \Rightarrow a-R < x < a+R$

4 possibilities for I: $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$

Example. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = \lim_{n\to\infty} \left| -3x \cdot \sqrt{\frac{n+1}{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| -3x \cdot \sqrt{\frac{1+1/n}{1+2/n}} \right| = |-3x|$$

By Ratio Test, the series converges when $|-3x| < 1$

$$|-3| \cdot |x| < 1$$

$$|x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$$

$$\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$$

$x = -\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{(-3)^n \cdot (-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3 \cdot (-\frac{1}{3}))^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$

Compare to $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ p-series $p = 1/2 < 1$ divergent but $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}} = 1 > 0$$

$\Rightarrow$ divergent by LCT

$x = \frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{(-3)^n \cdot (\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3 \cdot \frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

AST $b_n = \frac{1}{\sqrt{n+1}}$: $\frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} \Rightarrow b_n$ is decreasing

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0 \Rightarrow$ convergent

$$I = \left(-\frac{1}{3}, \frac{1}{3}\right]$$

Example. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{2(n+1)-1}\cdot\frac{2n-1}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x\cdot(2n-1)}{2n+2-1}\right| = \lim_{n\to\infty}\left|\frac{x\cdot(2n-1)}{2n+1}\right|$$

$$= |x|\cdot\lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\right| \overset{H}{=} |x|\cdot\lim_{n\to\infty}\left|\frac{2+0}{2+0}\right| = |x|\cdot 1 = |x|$$

$R = 1$

$|x| < 1$

$-1 < x < 1$

$x=-1$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$ AST $b_n = \frac{1}{2n-1}$ $\frac{1}{2(n+1)-1} < \frac{1}{2n-1}$ $\Rightarrow$ decreasing

$\lim_{n\to\infty}\frac{1}{2n-1} = 0$ $\Rightarrow$ convergent

$x=1$: $\sum_{n=1}^{\infty}\frac{1^n}{2n-1} = \sum_{n=1}^{\infty}\frac{1}{2n-1}$

Compare to $\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{1}{n}$ p-series $p=1$ divergent

$\frac{1}{2n-1} > \frac{1}{2n}$ $\Rightarrow$ $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges by the Comparison Test

$I = [-1, 1)$

Example. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$.

We apply the Root Test.

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|} = \lim_{n\to\infty} \left|\frac{x-2}{n}\right| = 0 < 1$

$\Rightarrow$ converges for all x

$R = \infty$

$I = (-\infty, \infty)$

Example. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-1)^{n+1}}{5^{n+1}\cdot\sqrt{n+1}} \cdot \frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty} \left|\frac{(2x-1)\cdot\sqrt{n}}{5\cdot\sqrt{n+1}}\right|$

$= \lim_{n\to\infty} \left|\frac{(2x-1)}{5}\cdot\sqrt{\frac{n}{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(2x-1)}{5}\cdot\sqrt{\frac{1}{1+1/n}}\right| = \frac{|2x-1|}{5}$

By Ratio Test, the series converges when $\frac{|2x-1|}{5} < 1$

$\Rightarrow |2x-1| < 5$

$\Rightarrow |2(x-1/2)| < 5$

$|x-1/2| < \frac{5}{2}$ ← R

$2x-1 = 2(x-1/2)$

$\Rightarrow (2x-1)^n = 2^n\cdot(x-1/2)^n$

$a = 1/2$

$\Rightarrow -5/2 < x-1/2 < 5/2$

$-2 < x < 3$

$\frac{3-(-2)}{2} \rightarrow$ $R = 5/2$

$x=-2$: $\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

AST $b_n = \frac{1}{\sqrt{n}}$ $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow$ dec

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ convergent

$x=3$: $\sum_{n=1}^{\infty} \frac{5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

p-series

$p = 1/2 < 1$

$\Rightarrow$ divergent

$I = [-2, 3)$

Homework 8.8 Solutions: Power Series

Prof. Leyda Almodóvar

Name: _____

1. (a) What is the radius of convergence of a power series? How do you find it?

   Given the series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is found by the Ratio Test and it is:

   i. 0 if the series converges only when $x = a$
   ii. ∞ if the series converges for all x, or
   iii. a positive number R such that the series converges if $|x-a| < R$.

   (b) What is the interval of convergence of a power series? How do you find it?

   This is the interval that consists of all values of x for which the series converges corresponding to the cases in part (a), the interval of convergence is

   i. the single point $\{a\}$
   ii. $(-\infty, \infty)$
   iii. $(a-R, a+R)$ [Note that we must the series for convergence at the endpoints $a-R$ and $a+R$]

2. Find the radius of convergence and interval of convergence of the series.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

   We apply the Ratio Test:

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{x\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| = \lim_{n\to\infty}\left|\frac{x\sqrt[3]{1}}{\sqrt[3]{1+\frac{1}{n}}}\right| = |x|$$

   Note that the series converges when $|x| < 1$. Hence, the radius of convergence $R = 1$. Now, we check the endpoints:

When $x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the Alternating Series Test since $b_n = \frac{1}{\sqrt[3]{n}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$.

When $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ is a divergent p-series with $p = \frac{1}{3} < 1$.

Hence, the interval of convergence is $(-1, 1]$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

We apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n x^n}\right| = \lim_{n\to\infty} \left|\frac{xn^2}{(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{x(1)}{(1+\frac{1}{n})^2}\right| = |x|$$

Note that the series converges when $|x| < 1$. Hence, the radius of convergence $R = 1$. Now, we check the endpoints:

When $x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges by the Alternating Series Test since $b_n = \frac{1}{n^2}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

When $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$.

Hence, the interval of convergence is $[-1, 1]$.

(c) $\sum_{n=1}^{\infty} n^n x^n$

We apply the Root Test:

$$\lim_{n\to\infty} |\sqrt[n]{a_n}| = \lim_{n\to\infty} \sqrt[n]{n^n x^n} = \lim_{n\to\infty} |nx| = \infty \text{ for all } x \neq 0$$

Hence, the radius of convergence is $R = 0$ and the interval of convergence is $\{0\}$.

Alternatively, we could apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1} x^{n+1}}{n^n x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1} x}{n^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(n+1)^n x}{n^n}\right|$$

$$= \lim_{n\to\infty} \left|(n+1)(1+\tfrac{1}{n})^n x\right| = \infty \text{ for all } x \neq 0$$

(d) $\sum_{n=1}^{\infty} 2^n n^2 x^n$

We apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)^2x^{n+1}}{2^n n^2 x^n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)^2 x}{n^2}\right| = \lim_{n\to\infty}\left|2(1+\tfrac{1}{n})^2 x\right| = 2|x|$$

The series converges when $2|x| < 1$, that is, when $|x| < \frac{1}{2}$. Hence, the radius of convergence $R = \frac{1}{2}$. Now, we check the endpoints:

When $x = -\frac{1}{2}$, $\sum_{n=1}^{\infty} 2^n n^2(-\frac{1}{2})^n = \sum_{n=1}^{\infty}(-1)^n n^2$ and when $x = \frac{1}{2}$,
$\sum_{n=1}^{\infty} 2^n n^2(\frac{1}{2})^n = \sum_{n=1}^{\infty} n^2$ and both series diverge by the Test for Divergence.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n5^n}$

We apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)5^{n+1}}\cdot\frac{n5^n}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\left|\frac{xn}{5(n+1)}\right| = \lim_{n\to\infty}\left|x(1+\tfrac{1}{n})\right| = \frac{|x|}{5}$$

The series converges when $\frac{|x|}{5} < 1$, that is, when $|x| < 5$. Hence, the radius of convergence is $R = 5$. We check the endpoints:

When $x = -5$, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-5)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n 5^n}{n5^n} = \sum_{n=1}^{\infty}\frac{-1}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent p-series since $p = 1$.

When $x = 5$, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(5)^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test since $b_n = \frac{1}{n}$ is decreasing and $\lim_{n\to\infty}\frac{1}{n} = 0$.

Thus, the interval of convergence is $(-5, 5]$.

(f) $\sum_{n=1}^{\infty}\frac{x^{2n}}{n!}$

We apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0 < 1 \text{ for all } x.$$

Hence, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

(g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$

We apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(2n+1)2^{n+1}}\cdot\frac{(2n-1)2^n}{(-1)^n(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)(2n-1)}{2(2n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-1)(2-\frac{1}{n})}{2(2+\frac{1}{n})}\right| = \frac{2|x-1|}{4}$$

The series converges when $\frac{|x-1|}{2} < 1$, that is, when $|x-1| < 2$, which is equivalent to $-2 < x-1 < 2$, which implies $-1 < x < 3$. Hence, the radius of convergence is $R = 2$. We check the endpoints:

When $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(-1-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by the Limit Comparison Test with $b_n = \frac{1}{n}$.

When $x = 3$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(3-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ converges by the Alternating Series Test since $b_n = \frac{1}{2n-1}$ is decreasing and $\lim_{t\to\infty} \frac{1}{2n-1} = 0$.

Thus, the interval of convergence is $(-1, 3]$.

(h) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n$

We apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}(x+6)^{n+1}}{8^{n+1}}\cdot\frac{8^n}{\sqrt{n}(x+6)^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n}}\cdot\frac{(x+6)}{8}\right|$$

$$= \lim_{n\to\infty}\left|\sqrt{1+\frac{1}{n}}\cdot\frac{(x+6)}{8}\right| = \frac{|x+6|}{8}$$

The power series converges when $\frac{|x+6|}{8} < 1$, that is $|x+6| < 8$, which is equivalent to $-8 < x+6 < 8$, which implies $-14 < x < 2$. Hence, the radius of convergence is $R = 8$. We check the endpoints:

When $x = -14$, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(-14+6)^n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(-8)^n = \sum_{n=1}^{\infty}(-1)^n\sqrt{n}$ diverges by the Test for Divergence.

When $x = 2$, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(2+6)^n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(8)^n = \sum_{n=1}^{\infty} \sqrt{n}$ diverges by the Test for Divergence.

Hence, the interval of convergence is $(-14, 2)$.

8.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. This strategy is useful for integrating functions that do not have elementary antiderivatives, solving differential equations, and for approximating functions by polynomials.

Recall $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$ and its sum is given by $\frac{a}{1-r} = \frac{1}{1-x}$.

We may express the function $f(x) = \frac{1}{1-x}$ as a power series:

$$(*) \qquad \boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x| < 1}$$

$a = x^0 = 1$
$r = x$

Example. Express $\frac{1}{1+x^3}$ as a power series and find the interval of convergence.

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n}}$$

Need $|-x^3| < 1$ $\quad (r = -x^3)$

$-1 < x^3 < 1$

$-1 < x < 1$

$\boxed{I = (-1, 1)}$

Example. Find a power series representation for $f(x) = \dfrac{3x^2}{1+x^3}$ and determine the interval of convergence.

$$\frac{3x^2}{1+x^3} = 3x^2 \cdot \frac{1}{1+x^3} = 3x^2 \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n} = 3 \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n} \cdot x^2$$

$$= \boxed{3 \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n+2}}$$

We found $I = (-1,1)$ in previous example. Multiplying by powers of x won't change I.

Example. Find a power series representation for $f(x) = \dfrac{x}{4-x}$ and determine its interval of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\frac{x}{4-x} = \frac{x}{4\left(1-\frac{x}{4}\right)} = \frac{x}{4} \cdot \frac{1}{1-\frac{x}{4}} = \frac{x}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$$

$$= \boxed{\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^{n+1}}$$

Need $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4 \Rightarrow -4 < x < 4$

$$\boxed{I = (-4,4)}$$

Differentiation and Integration of Power Series

Theorem. Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 \dots$ and let R be its radius of convergence. Then,

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

Note: The first term of f is a constant, so the index of f' goes up by 1.

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R. The interval of convergences may be different.

Example. Express $\frac{1}{(1-x)^2}$ as a power series by differentiating Equation (*). What is the radius of convergence?

$(*)\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1 \Leftarrow R=1$

$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$

$\frac{d}{dx}(1-x)^{-1} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$

$-(1-x)^{-2}\cdot(-1) = \sum_{n=1}^{\infty} nx^{n-1}$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \qquad R=1$$

Example. Find a power series representation for $\ln(4-x)$ and its radius of convergence.

Note $\ln(4-x)$ doesn't resemble $\frac{1}{1-x}$ at all!

$f(x)=\ln(4-x)$

$$f'(x)=\frac{1}{4-x}\cdot(-1)=\frac{-1}{4}\cdot\frac{1}{1-\frac{x}{4}}=-\frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n=-\sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}$$

$$\left|\frac{x}{4}\right|<1 \Rightarrow |x|<4$$

$R=4$

$$f(x)=\int f'(x)\,dx=\int -\sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}\,dx$$

$$\ln(4-x)=C-\sum_{n=0}^{\infty}\frac{x^{n+1}}{4^{n+1}\cdot(n+1)}$$

If $x=0$, $\ln 4=C$

$$\boxed{\ln(4-x)=\ln 4-\sum_{n=0}^{\infty}\frac{x^{n+1}}{4^{n+1}\cdot(n+1)}} \qquad R=4$$

Example. Find a power series representation for $f(x) = \tan^{-1} x$.

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$|-x^2| < 1 \Rightarrow |x| < 1 \quad R = 1$

$$f(x) = \int f'(x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$\tan^{-1} x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

If $x = 0$: $\tan^{-1}(0) = C + 0 \Rightarrow C = 0$

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}} \quad R = 1$$

Homework 8.9 Solutions: Representations of Functions as Power Series

Prof. Leyda Almodóvar

Name: _____

1. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the series $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$? Why?

   The series $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ also converges for $|x| < 2$ since we obtained it by integrating $\sum_{n=0}^{\infty} b_n x^n$.

2. Find a power series representation for the function and determine the interval of convergence.

   (a) $f(x) = \frac{5}{1-4x^2}$

   $f(x) = \frac{5}{1-4x^2} = 5 \cdot \frac{1}{1-(4x^2)} = 5 \sum_{n=0}^{\infty} (4x^2)^n = \boxed{\sum_{n=0}^{\infty} 5 \cdot 4^n x^{2n}}$

   This series converges when $|4x^2| < 1$, that is, $x^2 < \frac{1}{4}$. Hence, $-\frac{1}{2} < x < \frac{1}{2}$. The interval of convergence is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$

   (b) $f(x) = \frac{4}{2x+3}$

   $f(x) = \frac{4}{2x+3} = 4 \cdot \frac{1}{3(1+\frac{2x}{3})} = \frac{4}{3} \cdot \frac{1}{1-(-\frac{2x}{3})} = \frac{4}{3} \sum_{n=0}^{\infty} \left(-\frac{2x}{3}\right)^n$

   $= \sum_{n=0}^{\infty} \frac{4}{3} \cdot \frac{(-2)^n x^n}{3^n} = \boxed{\sum_{n=0}^{\infty} \frac{4(-2)^n x^n}{3^{n+1}}}$

The series converges when $\left|\frac{2x}{3}\right| < 1$, that is, $|x| < \frac{3}{2}$. Hence, the interval of convergence is $\boxed{\left(-\frac{3}{2}, \frac{3}{2}\right)}$

(c) $f(x) = \frac{x}{2x^2+1}$

$f(x) = \frac{x}{2x^2+1} = x \cdot \frac{1}{1-(2x^2)} = x \sum_{n=0}^{\infty} (-2x^2)^n = \boxed{\sum_{n=0}^{\infty} (-2)^n x^{2n+1}}$

The series converges when $|-2x^2| < 1$, that is $x^2 < \frac{1}{2}$. Hence, $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

The interval of convergence is $\boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$

3. Find a power series representation for the function and determine the radius of convergence.

(a) $f(x) = x^2 \tan^{-1}(x^3)$

We found $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ in class. Then,

$x^2 \tan^{-1}(x^3) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+5}}{2n+1}}$

The radius of convergence is $\boxed{R = 1}$

(b) $f(x) = \left(\frac{x}{2-x}\right)^3$

$f(x) = \left(\frac{x}{2-x}\right)^3 = \frac{x^3}{(2-x)^3} = x^3 \cdot \frac{1}{(2-x)^3}$

$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$

$\left(\frac{1}{2-x}\right)' = \frac{1}{(2-x)^2} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} n x^{n-1}$

$\left(\frac{1}{(2-x)^2}\right)' = \frac{2}{(2-x)^3} = \sum_{n=2}^{\infty} \frac{1}{2^{n+1}} n(n-1) x^{n-2} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}} x^n$

Note that n was replaced by $n+2$ to start the series at $n = 0$ instead of $n = 2$

Then, $\frac{1}{(2-x)^3} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+3}} x^n = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} x^n$ and

$f(x) = x^3 \cdot \frac{1}{(2-x)^3} = x^3 \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} x^n = \boxed{\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} x^{n+3}}$

The series converges when $\left|\frac{x}{2}\right| < 1$, that is, $|x| < 2$. Hence, the radius of convergence is $\boxed{R = 2}$

4. Evaluate the indefinite integral as a power series. What is the radius of convergence?

(a) $\int \frac{t}{1+t^3}\,\mathrm{d}t$

$\frac{t}{1+t^3} = t \cdot \frac{1}{1-(-t^3)} = t \sum_{n=0}^{\infty} (-t^3)^n = \sum_{n=0}^{\infty} (-1)^n t^{3n+1}$

$\int \frac{t}{1+t^3}\,\mathrm{d}t = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}}$

The series has the same radius of convergence as the series $\sum_{n=0}^{\infty} (-t^3)^n$, that is, $|-t^3| < 1$, which implies $|t| < 1$ and $\boxed{R = 1}$

(b) $\int \frac{\tan^{-1} x}{x}\,\mathrm{d}x$

We know $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Then,

$\frac{\tan^{-1} x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}$ and

$\int \frac{\tan^{-1} x}{x}\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}$

8.10 Taylor and Maclaurin Series

In the previous section we started looking at writing down a power series representation of a function. The problem is that there are functions that simply cannot be manipulated to resemble $f(x) = \dfrac{1}{1-x}$.

To come up with a more general method for writing a power series representation for a function we will make two assumptions.

1. First, let's assume that the function $f(x)$ has a power series representation about $x = a$: That is, $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$
2. Next, we assume that $f(x)$ has derivatives of every order and that we can find them.

We need to determine what the coefficients c_n are:

Note that if $x=a$, $f(a)=c_0$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$

$\Rightarrow f'(a) = c_1$

$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots$

$f''(a) = 2c_2 \Rightarrow c_2 = \dfrac{f''(a)}{2}$

$f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \ldots$

$f'''(a) = 3\cdot 2c_3 \Rightarrow c_3 = \dfrac{f'''(a)}{3\cdot 2}$

$f^{(4)}(x) = 4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5(x-a) + \ldots$

$f^{(4)}(a) = 4\cdot 3\cdot 2c_4 \Rightarrow c_4 = \dfrac{f^{(4)}(a)}{4\cdot 3\cdot 2}$

$$c_n = \frac{f^{(n)}(a)}{n!} \quad n \geq 0$$

Note: $0! = 1$

$n! = n\cdot(n-1)!$

$1! = 1\cdot 0!$

Taylor Series

If it exists, the **Taylor Series for** $f(x)$ **about** $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

for $|x-a| < R$ for some $R > 0$.

Maclaurin Series

If we use $a = 0$, we call the series a **Maclaurin Series** for $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$\Rightarrow a=0$

Example. Find the Maclaurin Series of $f(x) = e^x$. Also find the associated radius of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$f'(x) = e^x$
$f''(x) = e^x$
$\vdots$
$f^{(n)}(x) = e^x$
$f^{(n)}(0) = e^0 = 1$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x \cdot n!}{(n+1)\cdot n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1$$

$\Rightarrow \boxed{R = \infty}$

Example. Find the Maclaurin series for $\sin x$. Also find the associated radius of convergence.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

$f^{(0)}(x) = f(x) = \sin x$ | $f(0) = \sin 0 = 0$
$f'(x) = \cos x$ | $f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$ | $f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$ | $f'''(0) = -\cos 0 = -1$
$f^{(4)}(x) = \sin x$ | $f^{(4)}(0) = \sin 0 = 0$

All terms with even derivatives are $= 0$

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty} \left|\frac{-x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{-x^2 (2n+1)!}{(2n+3)(2n+2)(2n+1)!}\right| = \lim_{n\to\infty} \left|\frac{-x^2}{(2n+3)(2n+2)}\right| = 0 < 1$$

$\boxed{R = \infty}$

Example. Find the Maclaurin series for $\cos x$. Also find the associated radius of convergence.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+1) x^{2n}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}} \qquad \boxed{R = \infty}$$

Example. Find the Maclaurin series for $x^2 \cos x$. Also find the associated radius of convergence.

$$x^2 \cos x = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n)!}} \qquad \boxed{R = \infty}$$

Example. Find the Maclaurin series for ~~$f(x) = 2^x$. Also find the~~ associated radius of convergence.

$$2^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n}$$

$f^{(0)}(x) = f(x) = 2^x$

$f'(x) = 2^x \cdot \ln 2$

$f''(x) = 2^x \cdot (\ln 2)^2$

$f'''(x) = 2^x \cdot (\ln 2)^3$

$\vdots$

$f^{(n)}(x) = 2^x \cdot (\ln 2)^n, \quad n \geq 0$

$f^{(n)}(0) = 2^0 \cdot (\ln 2)^n = (\ln 2)^n$

$$\lim_{n\to\infty} \left| \frac{(\ln 2)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(\ln 2)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(\ln 2) x \cdot n!}{(n+1) \cdot n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(\ln 2) x}{n+1} \right| = 0 < 1$$

$\boxed{R = \infty}$

Example. Find the Taylor Series of $f(x) = \ln x$ about $a = 3$. Also find the associated radius of convergence.

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \frac{f^{(0)}(3)(x-3)^0}{0!} + \sum_{n=1}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$

$$f^{(0)}(x) = f(x) = \ln x$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = -\frac{1}{x^2} = -x^{-2}$$

$$f'''(x) = \frac{2}{x^3} = 2x^{-3}$$

$$f^{(4)}(x) = \frac{-2\cdot 3}{x^4} = -2\cdot 3\, x^{-4}$$

$$f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{x^5} = 2\cdot 3\cdot 4\, x^{-5}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}, \quad n \geq 1$$

$$f^{(n)}(3) = \frac{(-1)^{n+1}(n-1)!}{3^n}, \quad n \geq 1$$

$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n \cdot n!}(x-3)^n$$

$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n \cdot n(n-1)!}(x-3)^n$$

$$= \boxed{\ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n \cdot n}(x-3)^n} \qquad a_n$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-3)^{n+1}}{3^{n+1}\cdot(n+1)} \cdot \frac{3^n \cdot n}{(-1)^{n+1}(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-(x-3)\cdot n}{3(n+1)} \right|$$

$$= \left|\frac{-(x-3)}{3}\right| \cdot \lim_{n\to\infty} \left|\frac{n}{n+1}\right|$$

$$\overset{H}{=} \left|\frac{-(x-3)}{3}\right| \cdot \lim_{n\to\infty} \frac{1}{1+0}$$

$$= \left|\frac{(x-3)}{3}\right| < 1 \quad < R$$

$$\Rightarrow |x-3| < 3$$

a (center)

$$\boxed{R = 3}$$

Example. Find the Taylor Series for $f(x) = x^5 + 2x^3 + x$, $a = 2$. Also find the associated radius of convergence.

$$x^5+2x^3+x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$f(x) = x^5+2x^3+x$ | $f(2) = 2^5+2\cdot 2^3+2 = 50$

$f'(x) = 5x^4+6x^2+1$ | $f'(2) = 5\cdot 16 + 6\cdot 4 + 1 = 105$

$f''(x) = 20x^3+12x$ | $f''(2) = 20\cdot 8 + 12\cdot 2 = 184$

$f'''(x) = 60x^2+12$ | $f'''(2) = 60\cdot 4 + 12 = 252$

$f^{(4)}(x) = 120x$ | $f^{(4)}(2) = 120(2) = 240$

$f^{(5)}(x) = 120$ | $f^{(5)}(2) = 120$

$f^{(n)}(x) = 0$ for $n \geq 6$

$$x^5+2x^3+x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{50}{0!}(x-2)^0 + \frac{105}{1!}(x-2)^1 + \frac{184}{2!}(x-2)^2$$

$$+ \frac{252}{3!}(x-2)^3 + \frac{240}{4!}(x-2)^4 + \frac{120}{5!}(x-2)^5$$

$$= \boxed{50 + 105(x-2) + 92(x-2)^2 + 42(x-2)^3 + 10(x-2)^4 + (x-2)^5}$$

$\boxed{R = \infty}$ (f is a polynomial

Example. Find the Taylor Series for $f(x) = e^{2x}$, $a = 3$. Also find the associated radius of convergence.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \boxed{\sum_{n=0}^{\infty} \frac{e^6 \cdot 2^n}{n!}(x-3)^n}$$

$f^{(0)}(x) = f(x) = e^{2x}$

$f'(x) = e^{2x} \cdot 2$

$f''(x) = e^{2x} \cdot 2^2$

$f'''(x) = e^{2x} \cdot 2^3$

$\vdots$

$f^{(n)}(x) = e^{2x} \cdot 2^n, \; n \geq 0$

$f^{(n)}(3) = e^6 \cdot 2^n$

$$\lim_{n\to\infty} \left| \frac{e^6 \cdot 2^{n+1}(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^6 \cdot 2^n (x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-3) \cdot n! \cdot 2}{(n+1) \cdot n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-3) \cdot 2}{n+1} \right| = 0 < 1$$

$$\boxed{R = \infty}$$

Homework 8.10 Solutions: Taylor and Maclaurin Series

Prof. Leyda Almodóvar

Name: _____

1. Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

   (a) $f(x) = \frac{1}{1+x}$, $a = 2$

   $f^{(0)}(x) = f(x) = \frac{1}{1+x}$

   $f'(x) = -\frac{1}{(1+x)^2}$, $f(2) = \frac{1}{3}$

   $f''(x) = \frac{2}{(1+x)^3}$, $f'(2) = -\frac{1}{9}$

   $f'''(x) = -\frac{2\cdot 3}{(1+x)^4}$, $f''(2) = \frac{2}{27}$

   $f^{(4)}(x) = \frac{2\cdot 3\cdot 4}{(1+x)^5}$, $f'''(2) = -\frac{6}{81}$

   Then,

   $$\sum_{n=0}^{3} \frac{f^{(n)}(2)}{n!}(x-2)^n = \frac{\frac{1}{3}}{0!}(x-2)^0 + \frac{-\frac{1}{9}}{1!}(x-2) + \frac{\frac{2}{27}}{2!}(x-2)^2 + \frac{-\frac{6}{81}}{3!}(x-2)^3$$

   $$= \boxed{\frac{1}{3} - \frac{1}{9}(x-2) + \frac{1}{27}(x-2)^2 - \frac{1}{81}(x-2)^3}$$

   (b) $f(x) = \ln x$, $a = 1$

   $f^{(0)}(x) = f(x) = \ln x$, $f(1) = 0$

   $f'(x) = \frac{1}{x}$, $f'(1) = 1$

   $f''(x) = -\frac{1}{x^2}$, $f''(1) = -1$

   $f'''(x) = \frac{2}{x^3}$, $f'''(1) = 2$

   $f^{(4)}(x) = -\frac{6}{x^4}$, $f^{(4)}(1) = -6$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{06}{4!}(x-1)^4$$

$$= \boxed{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}$$

2. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

   (a) $f(x) = \ln(1+x)$

   $f^{(0)}(x) = \ln(1+x)$

   $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

   $f''(x) = -(1+x)^{-2}$

   $f'''(x) = 2(1+x)^{-3}$

   $f^{(4)}(x) = -6(1+x)^{-4}$

   In general, $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$, for $n \neq 0$. Note that $f^{(0)}(0) = \ln 1 = 0$. Then,

   $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(n-1)!}{n!}x^n = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n}$$

   We apply the Ratio Test to find the radius of convergence:

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{1+\frac{1}{n}}\right| = |x|$$

   Hence, the radius of convergence is $\boxed{R=1}$

   (b) $f(x) = e^{-2x}$

   $f^{(0)}(x) = e^{-2x}$

   $f'(x) = -2e^{-2x}$

   $f''(x) = 4e^{-2x}$

   $f'''(x) = -8e^{-2x}$

   $f^{(4)}(x) = 16e^{-2x}$

   In general, $f^{(n)}(x) = (-1)^n 2^n e^{-2x}$ and $f^{(n)}(0) = (-1)^n 2^n e^0 = (-2)^n$. Then,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}x^n}$$

We apply the Ratio Test to find the radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{(-2)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{-2x}{n+1}\right| = 0 < 1 \text{ for all } x$$

Hence, the radius of convergence is $\boxed{R=\infty}$

(c) $f(x) = x\cos x$

$f'(x) = -x\sin x + \cos x,\ f'(0) = 1$

$f''(x) = -x\cos x - 2\sin x,\ f''(0) = 0$

$f'''(x) = x\sin x - 3\cos x,\ f'''(0) = -3$

$f^{(4)}(x) = x\cos x + 4\sin x,\ f^{(4)}(0) = 0$

$f^{(5)}(x) = -x\sin x + 5\cos x,\ f^{(5)}(0) = 5.$

$f^{(6)}(x) = -x\cos x - 6\sin x,\ f^{(6)}(0) = 0$

$f^{(7)}(x) = x\sin x - 7\cos x,\ f^{(5)}(0) = -7.$

$$x\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \ldots$$

$$= 0 + (1)x + 0 - \frac{3}{3!}x^3 + 0 + \frac{5}{5!}x^5 + 0 - \frac{7}{7!}x^7 + \ldots = x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \frac{1}{6!}x^7 + \ldots$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n+1}}$$

Alternatively, we found in class that $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n}$, then

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n+1} \text{ and } \boxed{R=\infty}$$

3. Find the Taylor series for $f(x)$ centered at the given value of a. Also find the associated radius of convergence.

(a) $f(x) = x^6 - x^4 + 2,\ a = -2$

$f^{(0)}(x) = f(x) = x^6 - x^4 + 2,\ f(-2) = 50$

$f'(x) = 6x^5 - 4x^3,\ f'(-2) = -160$

$f''(x) = 30x^4 - 12x^2,\ f''(-2) = 432$

$f'''(x) = 120x^3 - 24x,\ f'''(-2) = -912$

$f^{(4)}(x) = 360x^2 - 24,\ f^{(4)}(-2) = 1416$

$f^{(5)(x)=720x},\ f^{(5)}(-2) = -1440$

$f^{(6)}(x) = 720,\ f^{(6)}(-2) = 720,$

$f^{(n)}(x) = 0$, for $n \geq 7$

$$x^6 - x^4 + 2 = \sum_{n=0}^{\infty} \frac{f^{(n)}(-2)}{n!}(x+2)^n = \frac{50}{0!}(x+2)^0 - \frac{160}{1!}(x+2)^1 + \frac{432}{2!}(x+2)^2 - \frac{912}{3!}(x+2)^3 + \frac{1416}{4!}(x+2)^4 - \frac{1440}{5!}(x+2)^5 + \frac{720}{6!}(x+2)^6 =$$

$$\boxed{50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3 + 59(x+2)^4 - 12(x+2)^5 + (x+2)^6}$$

A finite series converges for all x, so $\boxed{R = \infty}$

(b) $f(x) = \frac{1}{x}$, $a = -3$

$f^{(0)}(x) = f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = -\frac{2\cdot 3}{x^4}$

$f^{(4)}(x) = \frac{2\cdot 3\cdot 4}{x^5}$

$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$, then $f^{(n)}(-3) = (-1)^n \frac{n!}{(-3)^{n+1}} = (-1)^n \frac{n!}{(-1)^{n+1}3^{n+1}} = \frac{-n!}{3^{n+1}}$ and

$$\frac{1}{x} = \sum_{n=0}^{\infty} f^{(n)}(-3) = (-1)^n \frac{n!}{(-3)^{n+1}}(x+3)^n = \frac{-\frac{1}{3}}{0!}(x+3)^0 + \frac{-\frac{1}{3^2}}{1!}(x+3) + \frac{-\frac{2}{3^3}}{2!}(x+3)^2 + \frac{-\frac{6}{3^4}}{3!}(x+3)^3 + \ldots = \sum_{n=0}^{\infty} -\frac{n!}{n!3^{n+1}}(x+3)^n = \boxed{-\sum_{n=0}^{\infty} \frac{(x+3)^n}{3^{n+1}}}$$

We apply the Ratio Test to find the radius of convergence:

$$\lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(x+3)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+3)}{3} \right| = \frac{|x+3|}{3} < 1 \text{ when } |x+3| < 3.$$

Hence, the radius of convergence is $\boxed{R = 3}$

(c) $f(x) = \cos x,\ a = \frac{\pi}{2}$

$f^{(0)}(x) = f(x) = \cos x,\ f^{(0)}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$f'(x) = -\sin x,\ f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$

$f''(x) = -\cos x,\ f''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0$

$f'''(x) = \sin x,\ f'''\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$f^{(4)}(x) = \cos x,\ f^{(4)}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x - \tfrac{\pi}{2}\right)^n =$$

$$\tfrac{-1}{1!}(x-\tfrac{\pi}{2}) + \tfrac{1}{3!}(x-\tfrac{\pi}{2})^3 + \tfrac{-1}{5!}(x-\tfrac{\pi}{2})^5 + \tfrac{1}{7!}(x-\tfrac{\pi}{2})^7 + \ldots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \frac{\pi}{2}\right)^{2n+1}}$$

We apply the Ratio Test to find the radius of convergence:

$$\lim_{n\to\infty} \left| \tfrac{(x-\frac{\pi}{2})^{2n+3}}{(2n+3)!} \cdot \tfrac{(2n+1)}{(x-\frac{\pi}{2})^{2n+1}} \right| = \lim_{n\to\infty} \left| \tfrac{(x-\frac{\pi}{2})^2}{(2n+3)(2n+2)} \right| = 0 < 1 \text{ for all } x.$$

Hence, the radius of convergence is $\boxed{R = \infty}$

9 Differential Equations

9.1 Modeling with Differential Equations

Mathematical models often take the form of a **differential equation**, that is, an equation that contains an unknown function and some of its derivatives.

Definition. A **differential equation** is an equation that contains an unknown function and or more of its derivatives.

Definition. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. Ex: $y''+2y=0$ has order 2 Ex: $y'-y=5$ ← order 1

Definition. A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Note: When we are asked to solve a differential equation, we need to find *all* possible solutions of the equation.

Example. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$. (LHS, RHS)

$y' = \frac{2}{3}e^x - 2e^{-2x}$

$LHS = y' + 2y = \frac{2}{3}e^x - 2e^{-2x} + 2\left(\frac{2}{3}e^x + e^{-2x}\right)$

$= \frac{2}{3}e^x - 2e^{-2x} + \frac{4}{3}e^x + 2e^{-2x} = \frac{6}{3}e^x = 2e^x = RHS$

→ family of functions (r is constant)

Example. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?

$y = e^{rx}$

$y' = e^{rx} \cdot r$

$y'' = e^{rx} \cdot r^2$

$2r^2e^{rx} + re^{rx} - e^{rx} = 0$

$e^{rx}(2r^2 + r - 1) = 0$

$e^{rx} \neq 0$ $\quad 2r^2 + r - 1 = 0$

$(2r-1)(r+1) = 0$

$2r - 1 = 0$ $\quad r + 1 = 0$

$2r = 1$ $\quad r = -1$

$r = 1/2$

$y = e^{1/2x}$ and $y = e^{-x}$ are solutions of $2y'' + y' - y = 0$.

c is constant

Example. Show that every member of the family of functions $y = \dfrac{1+ce^t}{1-ce^t}$ is a solution of the differential equation $y' = \dfrac{1}{2}(y^2-1)$.

$$y' = \frac{(1+ce^t)'(1-ce^t) - (1-ce^t)'(1+ce^t)}{(1-ce^t)^2} = \frac{ce^t(1-ce^t) - (-ce^t)(1+ce^t)}{(1-ce^t)^2}$$

$$= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1-ce^t)^2} = \frac{2ce^t}{(1-ce^t)^2} = \text{LHS}$$

$$\text{RHS} = \frac{1}{2}(y^2-1) = \frac{1}{2}\left[\left(\frac{1+ce^t}{1-ce^t}\right)^2 - 1\right] = \frac{1}{2}\left[\frac{(1+ce^t)^2 - (1-ce^t)^2}{(1-ce^t)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1+2ce^t+c^2e^{2t} - (1-2ce^t+c^2e^{2t})}{(1-ce^t)^2}\right] = \frac{1}{2}\left[\frac{4ce^t}{(1-ce^t)^2}\right] = \frac{2ce^t}{(1-ce^t)^2}$$

LHS = RHS

In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial value problem (IVP)**.

Example. Find a solution of the differential equation $y' = \dfrac{1}{2}(y^2-1)$ that satisfies the initial condition $y(0) = 2$.

We know $y = \dfrac{1+ce^t}{1-ce^t}$ is a solution of the diff. eq.

$t=0, y=2$ $\quad 2 = \dfrac{1+ce^0}{1-ce^0} \Rightarrow 2 = \dfrac{1+c}{1-c} \Rightarrow 2(1-c) = 1+c$

$2-2c = 1+c$

$1 = 3c$

$\dfrac{1}{3} = c$

$$y = \frac{1+\frac{1}{3}e^t}{1-\frac{1}{3}e^t} \cdot \frac{3}{3} = \boxed{\frac{3+e^t}{3-e^t}}$$

Example. The Fitzhugh-Nagumo model for the electrical impulse in a neuron states that, in the absence of relaxation effects, the electrical potential in a neuron $v(t)$ obeys the differential equation $\frac{dv}{dt} = -v[v^2 - (1+a)v + a]$ where a is a positive constant such that $0 < a < 1$.

(a) For what values of v is v unchanging (that is, $\frac{dv}{dt} = 0$)?

$\frac{dv}{dt} = -v[v^2-(1+a)v+a] = 0$

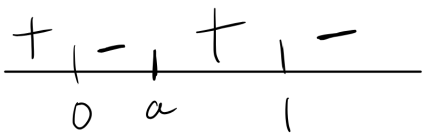
$-v(v-1)(v-a) = 0$

$v=0$ $\quad v-1=0$ $\quad v-a=0$

$v=1$ $\quad v=a$

(b) For what values of v is v increasing?

$\frac{dv}{dt} > 0$



$-v(v-1)(v-a)$

−	−	−	−
−	+	−	−
−	+	−	+
−	+	+	+

$v<0$

$0<v<a$

$a<v<1$

$v>1$

v is increasing when $v<0$ and $a<v<1$

(c) For what values of v is v decreasing?

v is decreasing when $0<v<a$ and when $v>1$.

Homework 9.1 Solutions: Modeling with Differential Equations

Prof. Leyda Almodóvar

Name: _____

1. Verify that $y = -t\cos t - t$ is a solution of the initial value problem $t\frac{dy}{dt} = y + t^2 \sin t$, $y(\pi) = 0$.

   We find $\frac{\mathrm{d}y}{\mathrm{d}t} = (-t)'\cos t - t(\cos t)' - (t)' = -\cos t + t\sin t - 1$

   Then, $t\frac{\mathrm{d}y}{\mathrm{d}t} = -t\cos t + t^2 \sin t - t = -t\cos t - t + t^2 \sin t = y + t^2 \sin t$

   Also, $y(\pi) = -\pi\cos(\pi) - \pi = -\pi(-1) - \pi = 0$ so the initial condition is satisfied.

2. (a) For what values of k does the function $y = \cos kt$ satisfy the differential equation $4y'' = -25y$?

   $y' = -k\sin(kt)$ and $y'' = -k^2\cos(kt)$. Hence,

   $4[-k^2\cos(kt)] = -25\cos(kt)$

   $-4k^2\cos(kt) = -25\cos(kt)$

   Hence, $4k^2 = 25$ and $k^2 = \frac{25}{4}$. Thus, $\boxed{k = \pm\frac{5}{2}}$

   (b) For those values of k, verify that every member of the family of functions $y = A\sin kt + B\cos kt$ is also a solution.

   $y' = Ak\cos(kt) - Bk\sin(kt)$ and $y'' = -Ak^2\sin(kt) - Bk^2\cos(kt)$

   The given equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Then, the left-hand side is equal to

   $$4y'' + 25y = 4[-Ak^2\sin(kt) - Bk^2\cos(kt)] + 25[Ak\cos(kt) - Bk\sin(kt)]$$
   $$= -4Ak^2\sin(kt) - 4Bk^2\cos(kt) + 25A\sin(kt) + 25B\cos(kt)$$
   $$= -4A\left(\tfrac{25}{4}\right)\sin\left(\pm\tfrac{5}{2}t\right) - 4B\left(\tfrac{25}{4}\right)\cos\left(\pm\tfrac{5}{2}t\right) + 25A\sin\left(\pm\tfrac{5}{2}t\right) + 25B\cos\left(\pm\tfrac{5}{2}t\right) = 0$$

Hence, every member of the family of functions $y = A\sin kt + B\cos kt$ is also a solution of the differential equation $4y'' = -25y$.

3. (a) Show that every member of the family of functions $y = \frac{\ln x + C}{x}$ is a solution of the differential equation $x^2y' + xy = 1$.

$y' = \frac{(\ln x + C)' \cdot x - (x)' \cdot (\ln x + C)}{x^2} = \frac{\frac{1}{x} \cdot x - (\ln x + C)}{x^2} = \frac{1 - \ln x - C}{x^2}$

The left-hand side is $x^2\left(\frac{1 - \ln x - C}{x^2}\right) + x\left(\frac{\ln x + C}{x}\right) = 1 - \ln x - C + \ln x + C = 1$

Hence, every member of the family of functions $y = \frac{\ln x + C}{x}$ is a solution of the differential equation $x^2y' + xy = 1$.

(b) Find a solution of the differential equation that satisfies the initial condition $y(1) = 2$.

$2 = \frac{\ln 1 + C}{1}$ implies that $C = 2$. Hence, $\boxed{y = \dfrac{\ln x + 2}{x}}$

(c) Find a solution of the differential equation that satisfies the initial condition $y(2) = 1$.

$1 = \frac{\ln 2 + C}{2}$ implies that $2 = \ln 2 + C$ so $C = 2 - \ln 2$.

Hence, $\boxed{y = \dfrac{\ln x + 2 - \ln 2}{x}}$

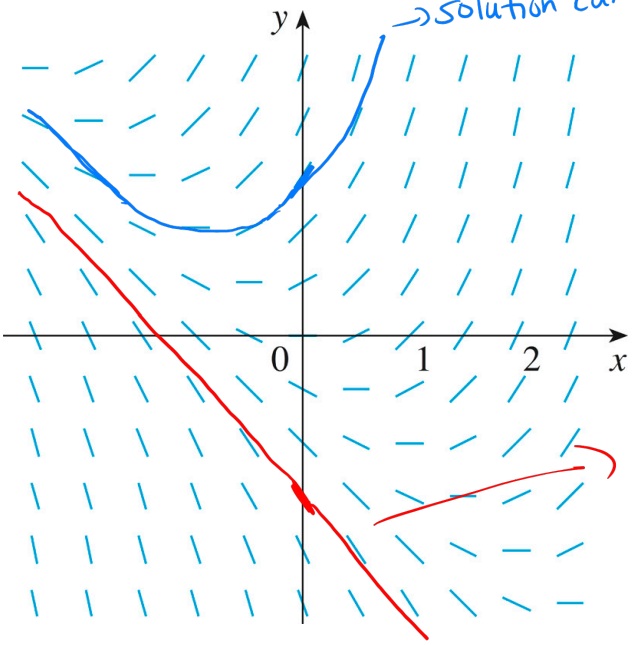
9.2 Direction Fields

Sometimes it's impossible to solve a differential equation in the sense of obtaining an explicit formula for the solution. We can still learn a lot about the solution through a graphical approach.

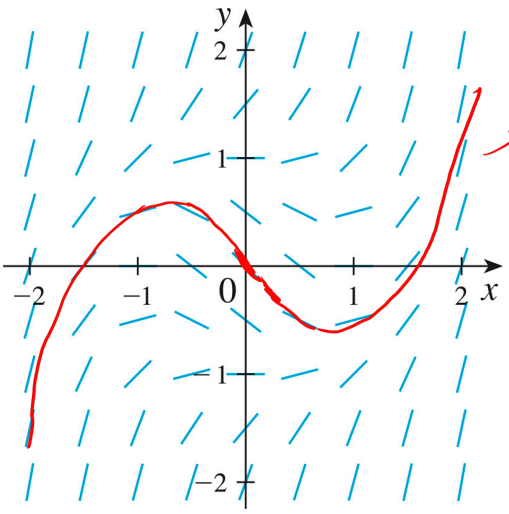
Example. Sketch the graph of the solution of the initial-value problem $y' = x + y$, $y(0) = 1$.

We don't know a formula for the solution so we have to try something else.

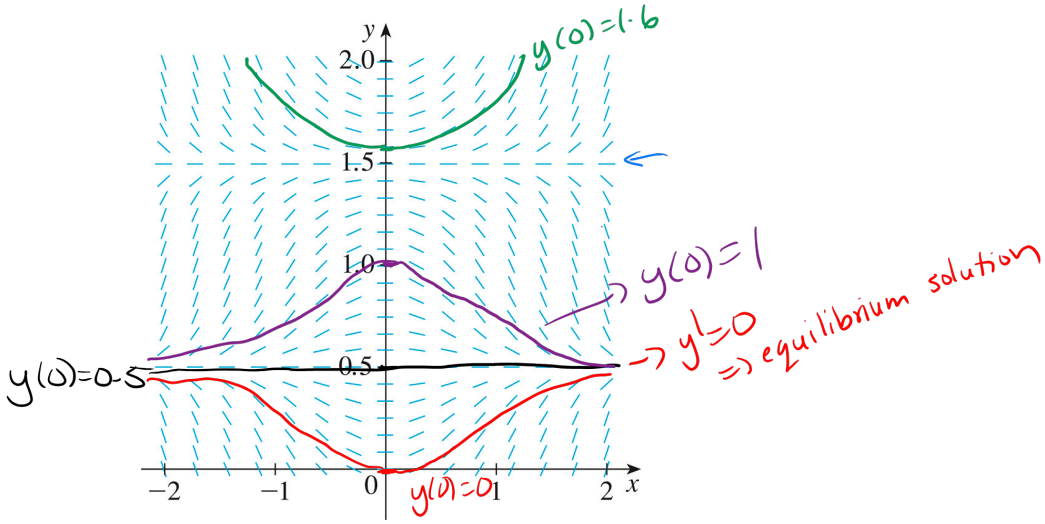
- $y' = x + y$ tells us that the slope at any point (x, y) on the graph (called the **solution curve**) is equal to the sum of the $x-$ and $y-$ coordinates of the point.
- The curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1.
- We construct a **direction field** by drawing short line segments at a number of points (x, y) with slope $x + y$.
- The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.



Example. The direction field for the differential equation $y' = x^2 + y^2 - 1$ is given. Sketch the solution curve that passes through the origin.



Example. A direction field for the differential equation $y' = x \cos \pi y$ is shown.



(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) $y(0) = 0$

(ii) $y(0) = 0.5$

(iii) $y(0) = 1$

(iv) $y(0) = 1.6$

(b) Find all the equilibrium solutions.

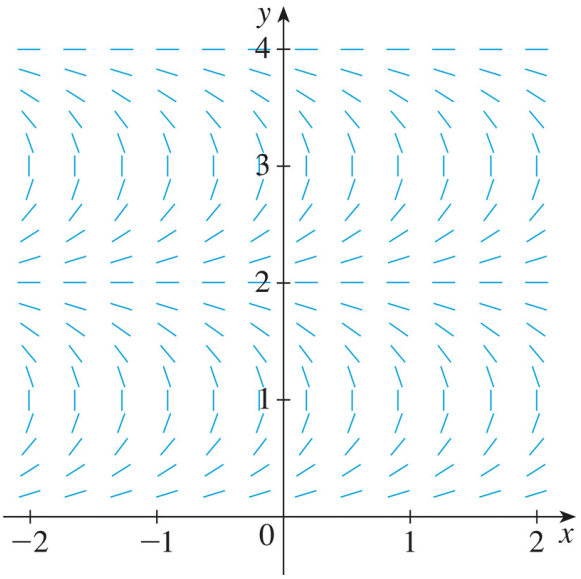
$y = 0.5$, $y = 1.5$

Homework 9.2 Solutions: Direction Fields

Prof. Leyda Almodóvar

Name: _____

1. A direction field for the differential equation $y' = \tan(\frac{1}{2}\pi y)$ is shown.



(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

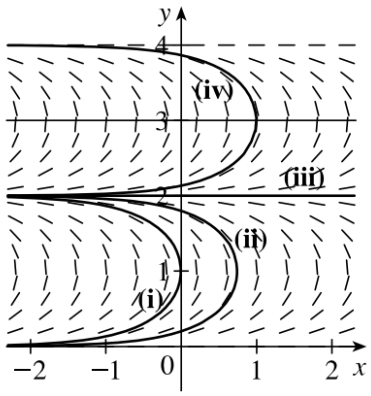
i. $y(0) = 1$

ii. $y(0) = 0.2$

iii. $y(0) = 2$

iv. $y(1) = 3$

The solutions are shown below:



(b) Find all the equilibrium solutions.

It appears that the constant functions $y = 0$, $y = 2$ and $y = 4$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = \tan(\frac{1}{2}\pi y)$.

2. Match the differential equation with its direction field (labeled I-IV). Give reasons for your answer.

(a) $y' = 2 - y$

The slopes at each point are independent of x, so the slopes are the same along each line parallel to the x-axis. Thus, III is the direction field for this equation. Also, note that for $y = 2$, $y' = 0$.

(b) $y' = x(2 - y)$

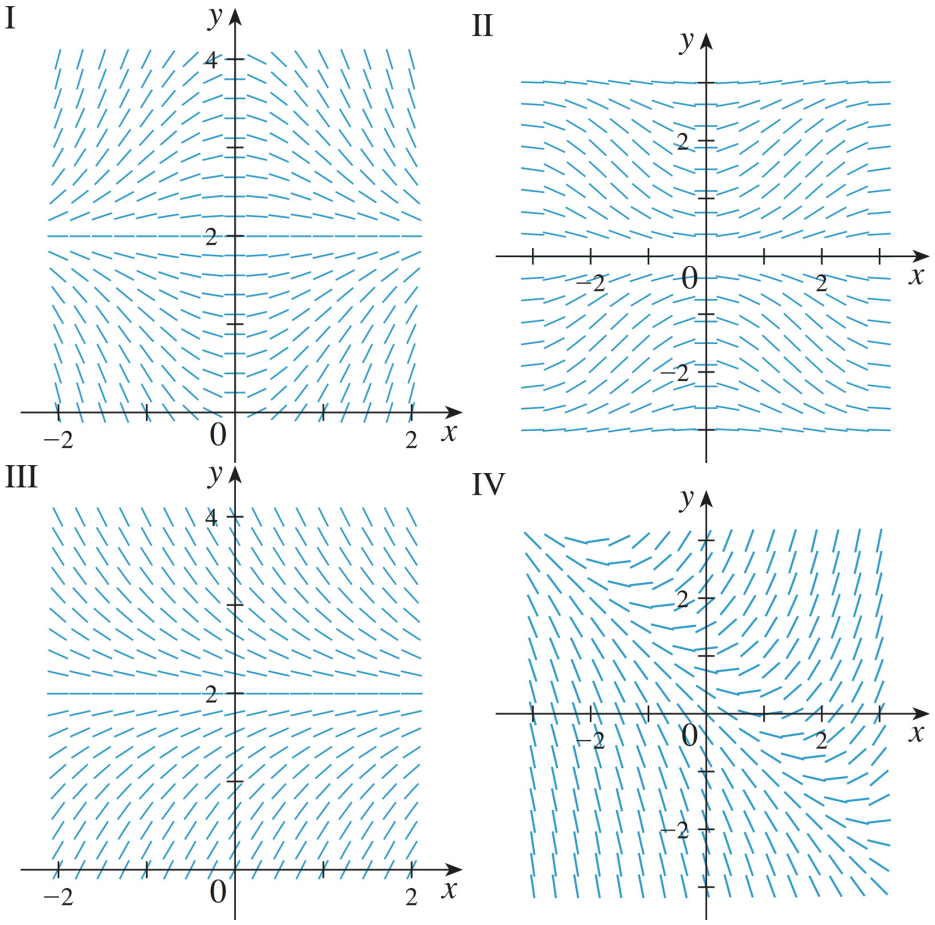
$y' = x(2 - y) = 0$ on the lines $x = 0$ and $y = 2$. Direction field I satisfies these conditions.

(c) $y' = x + y - 1$

$y' = x + y - 1 = 0$ on the line $y = -x + 1$. Direction field IV satisfies this condition. Notice also that on the line $y = -x$ we have $y = -1$, which is true in IV.

(d) $y' = \sin x \sin y$

$y' = \sin x \sin y = 0$ on the lines $x = 0$ and $y = 0$, and $y' > 0$ for $0 < x < \pi$, $0 < y < \pi$. Direction field II satisfies these conditions.

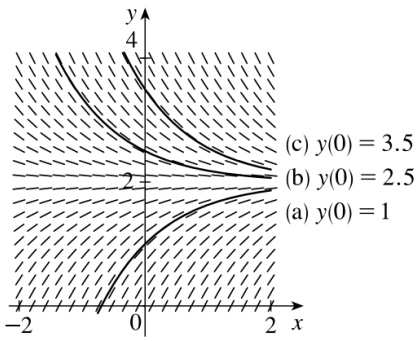

I
y
4
2
−2
0
2
x
II
y
2
−2
0
2
x
−2
III
y
4
2
−2
0
2
x
IV
y
2
−2
0
2
x
−2

3. Use the direction field labeled III above to sketch the graphs of the solutions that satisfy the given initial conditions.

   (a) $y(0) = 1$

   (b) $y(0) = 2.5$

   (c) $y(0) = 3.5$



9.3 Separable Equations

In this section we examine a certain type of differential equation that can be solved explicitly.

Definition. A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. In other words, $\frac{dy}{dx} = g(x) \cdot f(y)$. If $f(y) \neq 0$, then $\frac{dy}{f(y)} = g(x)\,dx$ and $\int \frac{dy}{f(y)} = \int g(x)\,dx$.

Example. Consider the following initial value problem:

(a) Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$.

$$y^2 dy = x^2 dx$$
$$\int y^2 dy = \int x^2 dx$$
$$\frac{y^3}{3} = \frac{x^3}{3} + C$$
$$y^3 = x^3 + 3C$$
$$\boxed{y = \sqrt[3]{x^3 + 3C}}$$

(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

$$x = 0, y = 2$$
$$\frac{2^3}{3} = \frac{0^3}{3} + C \Rightarrow C = \frac{8}{3}$$
$$y = \sqrt[3]{x^3 + 3 \cdot \frac{8}{3}} = \boxed{\sqrt[3]{x^3 + 8}}$$

Example. Solve the differential equation $\frac{dy}{dx} = \frac{5x^3}{3y + \sin y}$.

$$\int (3y + \sin y)\,dy = \int 5x^3 dx$$
$$\boxed{\frac{3y^2}{2} - \cos y = \frac{5x^4}{4} + C}$$

Example. Solve the equation $y' = x^2 y$.

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{y} = x^2 dx$$

$$\int \frac{1}{y}\, dy = \int x^2 dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

$$e^{\ln|y|} = e^{\frac{x^3}{3}+C}$$

$$|y| = e^{\frac{x^3}{3}+C}$$

$$\boxed{y = \pm e^{\frac{x^3}{3}+C}}$$

$$|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$$

Example. Solve the differential equation $xyy' = x^2 + 1$.

$$xy\frac{dy}{dx} = x^2 + 1$$

$$y\frac{dy}{dx} = \frac{x^2+1}{x}$$

$$y\,dy = \frac{x^2+1}{x}\,dx$$

$$\int y\,dy = \int x + \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \ln|x| + C$$

$$y^2 = x^2 + 2\ln|x| + 2C$$

$$\boxed{y = \pm\sqrt{x^2 + 2\ln|x| + 2C}}$$

Example. Solve the differential equation $\dfrac{dH}{dR} = \dfrac{RH^2\sqrt{1+R^2}}{\ln H}$.

$$\int \frac{\ln H}{H^2}\,dH = \int R\sqrt{1+R^2}\,dR$$

$u = \ln H \quad dv = \frac{1}{H^2}dH$

$du = \frac{1}{H}dH \quad v = -\frac{1}{H}$

$u = 1+R^2$

$du = 2R\,dR$

$\frac{1}{2}du = R\,dR$

$$-\frac{1}{H}\cdot\ln H + \int \frac{1}{H^2}\,dH = \frac{1}{2}\int \sqrt{u}\,du$$

$$-\frac{\ln H}{H} - \frac{1}{H} = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$-\frac{\ln H}{H} - \frac{1}{H} = \frac{1}{3}(1+R^2)^{3/2} + C$$

Example. Find the solution of the differential equation $\frac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$.

k is constant

$t=1,\ L=-1$

$$\frac{dL}{dt} = kL^2 \ln t$$

$$\int \frac{dL}{L^2} = \int k \ln t \, dt$$

$$-\frac{1}{L} = k(t\ln t - t) + C \leftarrow$$

$$\frac{-1}{k(t\ln t - t) + C} = L$$

$$\frac{-1}{-1} = k(1\cdot\ln 1 - 1) + C$$

$$1 = k(0-1) + C \Rightarrow C = 1 + k$$

$$L = \frac{-1}{k(t\ln t - t) + 1 + k} = \boxed{\frac{-1}{k(t\ln t - t + 1) + 1}}$$

Example. Find the solution of the differential equation $\frac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$.

$$\frac{dp}{dt} = t^2 p - p + t^2 - 1$$
$$= t^2 p + t^2 - p - 1$$
$$= t^2(p+1) - (p+1)$$
$$= (p+1)(t^2-1)$$

$$\frac{1}{p+1}\,dp = (t^2-1)\,dt$$
$$\int \frac{1}{p+1}\,dp = \int t^2 - 1\,dt$$
$$\ln|p+1| = \frac{t^3}{3} - t + C$$
$$|p+1| = e^{\frac{t^3}{3} - t + C}$$
$$p+1 = \pm e^{t^3/3 - t + C}$$
$$p = -1 \pm e^{t^3/3 - t + C}$$

Example. Find the solution of the differential equation $\frac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$.

$$\frac{d\theta}{dt} = \frac{t\sec\theta}{\theta e^{t^2}}$$

$$\frac{\theta}{\sec\theta}\,d\theta = \frac{t}{e^{t^2}}\,dt$$

$$\int \theta\cos\theta\,d\theta = \int \frac{t}{e^{t^2}}\,dt \qquad \eta = te^{-t^2}$$

$u = \theta \quad dv = \cos\theta\,d\theta$ | $u = -t^2$

$du = d\theta \quad v = \sin\theta$ | $du = -2t\,dt$

$-\frac{1}{2}du = t\,dt$

$$\theta\sin\theta - \int \sin\theta\,d\theta = -\frac{1}{2}\int e^u\,du$$

$$\theta\sin\theta - (-\cos\theta) = -\frac{1}{2}e^u + C$$

$$\boxed{\theta\sin\theta + \cos\theta = -\frac{1}{2}e^{-t^2} + C}$$

Example. Find the solution of the differential equation $\frac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$.

$\frac{dy}{dx} = xe^y$, $y(0)=0$

$\frac{1}{e^y}\,dy = x\,dx$

$x=0,\ y=0$

$\int e^{-y}\,dy = \int x\,dx$

$-e^0 = \frac{0^2}{2} + C$

$-e^{-y} = \frac{x^2}{2} + C$ ←

$-1 = C$

$u = -y$

$du = -dy$

$e^{-y} = -\frac{x^2}{2} - C$

$-y = \ln\left(-\frac{x^2}{2} - C\right)$

$y = -\ln\left(-\frac{x^2}{2} - C\right)$

$\boxed{y = -\ln\left(-\frac{x^2}{2} + 1\right)}$

Homework 9.3 Solutions: Separable Equations

Prof. Leyda Almodóvar

Name: _____

1. Solve the differential equation.

   (a) $\frac{dy}{dx} = x\sqrt{y}$

   $\frac{dy}{\sqrt{y}} = x\,dx$

   $\int \frac{dy}{\sqrt{y}} = \int x\,dx$

   $\int y^{-\frac{1}{2}}\,dy = \frac{x^2}{2} + C$

   $2y^{\frac{1}{2}} = \frac{x^2}{2} + C$

   $\sqrt{y} = \frac{x^2}{4} + \frac{C}{2}$

   $$\boxed{y = \left(\frac{x^2}{4} + \frac{C}{2}\right)^2}$$

   (b) $y' + xe^y = 0$

   $\frac{dy}{dx} + xe^y = 0$

   $\frac{dy}{dx} = -xe^y$

   $\int \frac{dy}{e^y} = \int -x\,dx$

   $-e^{-y} = -\frac{x^2}{2} + C$

   $e^{-y} = \frac{x^2}{2} - C$

   $-y = \ln\left(\frac{x^2}{2} - C\right)$

   $$\boxed{y = -\ln\left(\frac{x^2}{2} - C\right)}$$

(c) $\frac{du}{dt} = \frac{1+t^4}{ut^2+u^4t^2}$

$\frac{du}{dt} = \frac{1+t^4}{t^2(u+u^4)}$

$(u+u^4)\,du = \frac{1+t^4}{t^2}\,dt$

$\int (u+u^4)\,du = \int \left(t^2 + \frac{1}{t^2}\right) dt$

$$\boxed{\frac{u^2}{2} + \frac{u^5}{5} = \frac{t^3}{3} - \frac{1}{t} + C}$$

(d) $\frac{dz}{dt} + e^{t+z} = 0$

$\frac{dz}{dt} = -e^{t+z}$

$\frac{dz}{dt} = -e^t e^z$

$\int \frac{dz}{e^z} = \int -e^t\,dt$

$-e^{-z} = -e^t + C$

$e^{-z} = e^t - C$

$-z = \ln(e^t - C)$

$$\boxed{z = -\ln(e^t - C)}$$

2. Find the solution of the differential equation that satisfies the given initial condition.

(a) $\frac{dy}{dx} = \frac{x \sin x}{y}$, $y(0) = -1$.

$\int y\,dy = \int x \sin x\,dx$

We apply integration by parts on the right-hand side with $u = x$, $du = dx$, $dv = \sin x\,dx$, and $v = -\cos x$:

$\frac{y^2}{2} = -x\cos x + \sin x + C$

We solve for C using the initial condition $x = 0$ and $y = -1$:

$\frac{(-1)^2}{2} = -0\cos(0) + \sin(0) + C$

$C = \frac{1}{2}$

Hence, $\frac{y^2}{2} = -x\cos x + \sin x + \frac{1}{2}$

$y^2 = -2x + 2\sin x + 1$

$y = \pm\sqrt{2\sin x - 2x\cos x + 1}$

Since $y(0) = -1 < 0$ we keep the negative solution

$$\boxed{y = -\sqrt{2\sin x - 2x\cos x + 1}}$$

(b) $x + 3y^2\sqrt{x^2+1}\frac{dy}{dx} = 0$, $y(0) = 1$.

$3y^2\sqrt{x^2+1}\frac{dy}{dx} = -x$

$3y^2\,dy = -\frac{x}{\sqrt{x^2+1}}\,dx$

$\int 3y^2\,dy = -\int \frac{x}{\sqrt{x^2+1}}\,dx$

We apply u-substitution on the right-hand side with $u = x^2+1$, $du = 2x\,dx$:

$y^3 = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

$y^3 = -u^{\frac{1}{2}} + C$

$y^3 = -(x^2+1)^{\frac{1}{2}} + C$

We solve for C using the initial condition $x = 0$ and $y = 1$:

$1 = -(0^2+1)^{\frac{1}{2}} + C$

$2 = C$

Hence, $y^3 = -\sqrt{x^2+1} + 2$

$$\boxed{y = \sqrt[3]{2 - \sqrt{x^2+1}}}$$

(c) $\frac{dP}{dt} = \sqrt{Pt}$, $P(1) = 2$.

$\frac{dP}{dt} = \sqrt{P}\sqrt{t}$

$\int \frac{dP}{\sqrt{P}} = \int \sqrt{t}\,dt$

$2\sqrt{P} = \frac{2}{3}t^{\frac{3}{2}} + C$

We solve for C using the initial condition $t = 1$ and $P = 2$:

$2\sqrt{2} = \frac{2}{3}(1)^{\frac{3}{2}} + C \qquad \Rightarrow \qquad 2\sqrt{2} - \frac{2}{3} = C$

Hence, $\sqrt{P} = \frac{1}{3}t^{\frac{3}{2}} + \sqrt{2} - \frac{1}{3}$

$$\boxed{P = \left(\frac{t^{\frac{3}{2}}}{3} + \sqrt{2} - \frac{1}{3}\right)^2}$$

9.4 Models for Population Growth

Law of Natural Growth

Assumption: The population $P(t)$ grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

K is constant

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C}$$

$$P(t) = e^{kt} \cdot e^{C}$$

$$\boxed{P(t) = P(0)e^{kt}}$$

Note: $P(0) = e^{0t} \cdot e^{C} = e^{C}$

↙ Not very realistic

*DO not use on homework Problems

The Logistic Model

Many populations start by increasing in an exponential manner, but the population levels off when it approaches its **carrying capacity** M (or decreases toward M if it ever exceeds M). For a more realistic model to take into account both trends, we make 2 assumptions:

- $\frac{dP}{dt} \approx kP$ if P is small since initially, the growth rate is proportional to P.
- $\frac{dP}{dt} < 0$ if $P > M$ since P decreases if it ever exceeds M.

The model is described by the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$ ✓separable

$$\int \frac{1}{P(1-P/M)}\,dP = \int k\,dt$$

$$\int \frac{M}{P(M-P)}\,dP = kt + C$$

$$\frac{M}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$M = A(M-P) + B\cdot P$$

$$M = \underbrace{A}_{=1} M + P\underbrace{(B-A)}_{=0}$$

$A = 1$, $A = B \Rightarrow B = 1$

$$\int \frac{1}{P} + \frac{1}{M-P}\,dP = kt + C$$

$$\ln|P| - \ln|M-P| = kt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kt + C$$

$$-\ln\left|\frac{P}{M-P}\right| = -kt - C \Rightarrow \ln\left|\left(\frac{P}{M-P}\right)^{-1}\right| = -kt - C$$

$$\Rightarrow \ln\left|\frac{M-P}{P}\right| = -kt - C \Rightarrow \left|\frac{M-P}{P}\right| = e^{-kt-C} \quad (*)$$

$$\frac{M-P}{P} = \pm e^{-kt-C} \Rightarrow \frac{M}{P} - 1 = \pm e^{-kt-C}$$

$$\Rightarrow \frac{M}{P} = 1 \pm e^{-kt-C} \Rightarrow \boxed{P(t) = \frac{M}{1 \pm e^{-kt}\cdot e^{-C}}}$$

Let $t=0$: $\dfrac{M-P(0)}{P(0)} = \pm e^{0}\cdot e^{-C}$

$$\boxed{\pm e^{-C} = \frac{M-P(0)}{P(0)}}$$

Example. A population is modeled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = 1.2P\left(1 - \frac{P}{4200}\right)$.

(a) For what values of P is the population increasing?

(b) For what values of P is the population decreasing?

(c) What are the equilibrium solutions?

(a) $\frac{dP}{dt} > 0 \Rightarrow 1.2P\left(1-\frac{P}{4200}\right) > 0$

$\Rightarrow P > 0$ and $1 - \frac{P}{4200} > 0 \Rightarrow 1 > \frac{P}{4200}$

$\Rightarrow \boxed{P < 4200}$

(b) $\frac{dP}{dt} < 0 \Rightarrow 1.2P\left(1-\frac{P}{4200}\right) < 0$

$\Rightarrow P > 0$ and $1 - \frac{P}{4200} < 0 \Rightarrow 1 < \frac{P}{4200}$

$\boxed{P > 4200}$

(c) $\frac{dP}{dt} = 0 \Rightarrow 1.2P\left(1-\frac{P}{4200}\right) = 0$

$\Rightarrow \boxed{P = 0}$ $1 = \frac{P}{4200} \Rightarrow \boxed{P = 4200}$

Compare $\frac{dP}{dt} = kP\left(1-\frac{P}{M}\right)$, $k = 1.2$, $M = 4200$

Logistic Model

Example. Write the solution of the IVP $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$, $P(0) = 100$ and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

Given: $M = 1000$
$K = 0.08$
$P(0) = 100$

$P(t) = \frac{M}{1 \pm e^{-C} e^{-Kt}}$, where $\pm e^{-C} = \frac{M - P(0)}{P(0)}$

$\pm e^{-C} = \frac{1000 - 100}{100} = \frac{900}{100} = 9$

$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$

$P(40) = \frac{1000}{1 + 9e^{-0.08(40)}} \approx 731$

$P(80) = \frac{1000}{1 + 9e^{-0.08(80)}} \approx 985$

$900 = \frac{1000}{1 + 9e^{-0.08t}} \Rightarrow 1 + 9e^{-0.08t} = \frac{1000}{900} \Rightarrow 9e^{-0.08t} = \frac{1}{9}$

$\Rightarrow e^{-0.08t} = \frac{1}{81} \Rightarrow -0.08t = \ln(1/81) \Rightarrow t = \frac{\ln(\frac{1}{81})}{-0.08}$

$t = \frac{\ln 1 - \ln 81}{-0.08} = \frac{\ln 81}{0.08} \approx 54.9$

Example. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after one year, what will the population be after another three years?

$P(t) = \frac{M}{1 \pm e^{-C} e^{-kt}}$, $\pm e^{-C} = \frac{M - P(0)}{P(0)}$

Given: $P(0) = 1000$, $M = 10000$, $P(1) = 2500$ Need: k

$\pm e^{-C} = \frac{10000 - 1000}{1000} = 9$

$P(t) = \frac{10000}{1 + 9e^{-kt}}$ $\quad 2500 = \frac{10000}{1 + 9e^{-k}}$

$1 + 9e^{-k} = \frac{10000}{2500} \Rightarrow 1 + 9e^{-k} = 4 \Rightarrow 9e^{-k} = 3$

$\Rightarrow e^{-k} = \frac{1}{3} \Rightarrow -k = \ln\left(\frac{1}{3}\right) \Rightarrow k = -\ln\left(\frac{1}{3}\right) = \ln 3$

$P(t) = \frac{10000}{1 + 9e^{-(\ln 3)t}}$

$P(4) = \frac{10000}{1 + 9e^{-(\ln 3)\cdot 4}} = \frac{10000}{1 + 9 \cdot [e^{\ln 3}]^{-4}} = \frac{10000}{1 + 9 \cdot 3^{-4}} = \frac{10000}{1 + 9/81} = \frac{10000}{1 + \frac{1}{9}}$

$= 10000 \cdot \frac{9}{10} = \boxed{9000}$

Homework 9.4 Solutions: Models for Population Growth

Prof. Leyda Almodóvar

Name: _____

1. A population grows according to the logistic equation $\frac{dP}{dt} = 0.02P - 0.0004P^2$, $P(0) = 40$, where t is measured in weeks.

   (a) What is the carrying capacity? What is the value of k?

   First, we rewrite the logistic equation in standard form:

   $$\frac{dP}{dt} = 0.02P - 0.0004P^2 = 0.02P(1 - 0.02P) = 0.02P\left(1 - \frac{P}{50}\right)$$

   Hence, the carrying capacity is $\boxed{M = 50 \text{ individuals}}$ and $\boxed{k = 0.02}$

   (b) Write the solution of the equation.

   $$\pm e^C = \frac{M - P(0)}{P(0)} = \frac{50-40}{40} = \frac{10}{40} = \frac{1}{4}$$

   $$P = \frac{M}{1 \pm e^{-C}e^{-kt}} = \frac{50}{1 + \frac{1}{4}e^{-0.02t}}$$

   Multiplying the top and bottom by 4 we obtain:

   $$\boxed{P = \frac{200}{4 + e^{-0.02t}} \text{ individuals}}$$

   (c) What is the population after 10 weeks?

   $$P(10) = \frac{200}{4+e^{-0.02(10)}} = \boxed{\frac{200}{4 + e^{-0.2}} \text{ individuals}}$$

2. Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2$, $P(0) = 50$ where t is measured in years.

   (a) What is the carrying capacity?

   First, we rewrite the logistic equation in standard form:

$\frac{dP}{dt} = 0.4P - 0.001P^2 = 0.4P(1 - 0.0025P) = 0.4P\left(1 - \frac{P}{400}\right)$

Hence, the carrying capacity is $\boxed{M = 400 \text{ individuals}}$

(b) What is $P'(0)$?

First we find $k = 0.4$, $\pm e^C = \frac{M - P(0)}{P(0)} = \frac{400-50}{50} = \frac{350}{50} = 7$ and $P(t) = \frac{M}{1 \pm e^{-C}e^{-kt}} = \frac{400}{1+7e^{-0.4t}}$. Then,

$P'(0) = kP(0)\left(1 - \frac{P(0)}{M}\right) = 0.4(50)\left(1 - \frac{50}{400}\right) = 20\left(1 - \frac{1}{8}\right) = \frac{140}{8} = \boxed{17.5 \text{ individuals per year}}$

(c) When will the population reach 50% of the carrying capacity?

50% of 400 is 200. Hence, $200 = \frac{400}{1+7e^{-0.4t}}$

$1 + 7e^{-0.4t} = 2$

$7e^{-0.4t} = 1$

$e^{-0.4t} = \frac{1}{7}$

$-0.4t = \ln\left(\frac{1}{7}\right)$

$\boxed{t = \dfrac{\ln 7}{0.4} \text{ years}}$

3. (a) Assume that the carrying capacity for the US population is 800 million. Use it and the fact that population was 282 million in 2000 to formulate a logistic model for the US population.

We are given that $M = 800$ million. In 2000 $t = 0$ and $P(0) = 282$ million. Hence,

$e^{-C} = \frac{800-282}{282} = \frac{259}{141}$

$P(t) = \frac{800}{1+\frac{259}{141}e^{-kt}} = \boxed{\dfrac{112,800}{141 + 259e^{-kt}} \text{ millions of people}}$

(b) Determine the value of k in your model by using the fact that the population in 2010 was 309 million.

In 2010 $t = 10$, $P(10) = 309$ million. Hence,
$309 = \frac{112,800}{141+259e^{-10k}}$

$141 + 259e^{-10k} = \frac{112{,}800}{309}$

$259e^{-10k} = \frac{112{,}800}{309} - 141$

$259e^{-10k} = \frac{69{,}231}{309}$

$e^{-10k} = \frac{69{,}231}{80{,}031} = \frac{23{,}077}{26{,}677}$

$-10k = \ln\left(\frac{23{,}077}{26{,}677}\right)$

$$\boxed{k = -\frac{1}{10}\ln\left(\frac{23,077}{26,677}\right)}$$

(c) Use your model to predict the US population in the years 2100 and 2200.

In 2100 $t = 100$ and in 2200 $t = 200$. Hence,

$$\boxed{P(100) = \frac{112,800}{141 + 259e^{10\ln(\frac{23,077}{26,677})}} \text{ millions of people}} \approx 559 \text{ millions}$$

$$\boxed{P(200) = \frac{112,800}{141 + 259e^{20\ln(\frac{23,077}{26,677})}} \text{ millions of people}} \approx 727 \text{ millions}$$

(d) Use your model to predict the year in which the US population will exceed 500 million.

$500 = \frac{112{,}800}{141+259e^{\frac{1}{10}\ln(\frac{23,077}{26,677})t}}$

$141 + 259e^{\frac{1}{10}\ln(\frac{23,077}{26,677})t = \frac{1{,}128}{5}}$

$259e^{\frac{1}{10}\ln(\frac{23,077}{26,677})t = \frac{423}{5}}$

$e^{\frac{1}{10}\ln(\frac{23,077}{26,677})t = \frac{423}{1{,}295}}$

$\frac{1}{10}\ln(\frac{23,077}{26,677})t = \ln\left(\frac{423}{1{,}295}\right)$

$$\boxed{t = 10\frac{\ln(\frac{423}{1,295})}{\ln(\frac{23,077}{26,677})} \text{ years}} \approx 77.18 \text{ years}$$

4. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.

   (a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

   We are given that $P(0) = 400$, $M = 10000$, and $P(1) = 3(400) = 1200$. Then, $e^{-C} = \frac{10000-400}{400} = \frac{9600}{400} = 24$ and $P(t) = \frac{10000}{1+24e^{-kt}}$

   We find k using $P(1) = 1200$:

   $1200 = \frac{10000}{1+24e^{-k}}$

   $1 + 24e^{-k} = \frac{10000}{1200}$

   $24e^{-k} = \frac{25}{3} - 1$

   $24e^{-k} = \frac{22}{3}$

   $e^{-k} = \frac{11}{36}$

   $-k = \ln\left(\frac{11}{36}\right)$

   $k = -\ln\left(\frac{11}{36}\right)$

   Hence,

   $$\boxed{P(t) = \frac{10000}{1 + 24e^{\ln(\frac{11}{36})t}} \text{ fish}}$$

   (b) How long will it take for the population to increase to 5000?

   $5000 = \frac{10000}{1+24e^{\ln(\frac{11}{36})t}}$

   $1 + 24e^{\ln(\frac{11}{36})t} = 2$

   $24e^{\ln(\frac{11}{36})t} = 1$

   $e^{\ln(\frac{11}{36})t} = \frac{1}{24}$

   $\ln\left(\frac{11}{36}\right)t = \ln\left(\frac{1}{24}\right)$

   $$\boxed{t = \frac{-\ln 24}{\ln(\frac{11}{36})} \text{ years}}$$

9.5 Linear Equations

Definition. A first-order **linear** differential equation is one that can be put into the form $\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$ where P and Q are continuous functions on a given interval.

A first-order linear differential equation can be solved by multiplying both sides of the equation $\frac{dy}{dx} + P(x)y = Q(x)$ by an **integrating factor** $I(x) = e^{\int P(x)\,dx}$ so that the left-hand side is $\frac{d}{dx}\Big(I(x)y\Big)$.

Example. Solve the differential equation $y' + 3x^2y = 6x^2$.

$P(x) = 3x^2 \Rightarrow I(x) = e^{\int 3x^2 dx} = e^{x^3 + C}$ *C can be anything so C=0

$\Rightarrow e^{x^3}y' + e^{x^3}\cdot 3x^2 y = 6x^2 \cdot e^{x^3}$

$\frac{d}{dx}(e^{x^3}y) = 6x^2 \cdot e^{x^3}$

$\int \frac{d}{dx}(e^{x^3}y)\,dx = \int 6x^2 e^{x^3}\,dx$

$u = x^3$, $du = 3x^2\,dx$, $\frac{1}{3}du = x^2\,dx$

$e^{x^3}\cdot y = 2\int e^u\,du$

$= 2e^u + C$

$= 2e^{x^3} + C$

$y = \frac{2e^{x^3}}{e^{x^3}} + \frac{K}{e^{x^3}} \Rightarrow \boxed{y = 2 + \frac{K}{e^{x^3}}}$

Example. Solve the differential equation $y' + y = 1$.

$P(x) = 1 \Rightarrow I(x) = e^{\int 1\,dx} = e^x$

$e^x y' + e^x y = e^x$

$\frac{d}{dx}(e^x y) = e^x$

$\int \frac{d}{dx}(e^x y)\,dx = \int e^x dx$

$e^x y = e^x + C$

$$\boxed{y = 1 + \frac{C}{e^x}}$$

Example. Find the solution of the IVP $x^2y' + xy = 1$, $x > 0$, $y(1) = 2$.

not in standard form

$x^2y' + xy = 1$

$y' + \frac{x}{x^2} \cdot y = \frac{1}{x^2}$

$y' + \frac{1}{x} \cdot y = \frac{1}{x^2}$

$\Rightarrow P(x) = \frac{1}{x} \Rightarrow I(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x|$

$x > 0 \Rightarrow = x$

$xy' + y = \frac{1}{x}$

$\frac{d}{dx}(xy) = \frac{1}{x}$

$xy = \int \frac{1}{x} dx$

$= \ln|x| + C$ ←

$x = 1, y = 2$

$1 \cdot 2 = \ln 1 + C$

$2 = C$

$y = \frac{\ln x}{x} + \frac{C}{x}$

$\boxed{y = \frac{\ln x}{x} + \frac{2}{x}}$

Example. Solve the differential equation $y' = x - y$.

$$y' + y = x$$

$$P(x) = 1 \Rightarrow I(x) = e^{\int 1\,dx} = e^x$$

$$e^x y' + e^x y = x e^x$$

$$\frac{d}{dx}(e^x y) = x e^x$$

$$e^x y = \int x e^x\,dx$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$e^x y = x e^x - \int e^x\,dx$$

$$e^x y = x e^x - e^x + C$$

$$\boxed{y = x - 1 + \frac{C}{e^x}}$$

$r \cdot \ln x = \ln x^r$

Example. Solve the differential equation $t^2 \dfrac{dy}{dt} + 3ty = \sqrt{1+t^2}$, $t > 0$.

$\dfrac{dy}{dt} + \dfrac{3}{t}y = \dfrac{\sqrt{1+t^2}}{t^2}$

$P(t) = \dfrac{3}{t} \Rightarrow I(t) = e^{\int \frac{3}{t}dt}$

$= e^{3\ln|t|}$

$= e^{\ln t^3}$

$= t^3$

$t^3 \dfrac{dy}{dt} + 3t^2 y = t\sqrt{1+t^2}$

$\dfrac{d}{dt}(t^3 y) = t\sqrt{1+t^2}$

$t^3 y = \int t\sqrt{1+t^2}\,dt$

$u = 1+t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$t^3 y = \frac{1}{2}\int \sqrt{u}\,du$

$t^3 y = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$= \frac{1}{3}(1+t^2)^{3/2} + C$

$$\boxed{y = \frac{(1+t^2)^{3/2}}{3t^3} + \frac{C}{t^3}}$$

$x > 0$

Example. Solve the initial-value problem $x^2y' + 2xy = \ln x$, $y(1) = 2$.

$y' + \frac{2}{x} \cdot y = \frac{\ln x}{x^2}$ $\quad P(x) = \frac{2}{x} \Rightarrow I(x) = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{2\ln x} = e^{\ln x^2} = x^2$

$x^2 y' + 2xy = \ln x$

$\frac{d}{dx}(x^2 y) = \ln x$

$x^2 y = \int \ln x \, dx$

$= x\ln x - x + C$ ←

$y = \frac{\ln x}{x} - \frac{1}{x} + \frac{C}{x^2}$

Forgot to find C in class

$x = 1, y = 2$

$1^2 \cdot 2 = -1 \cdot \ln 1 - 1 + C$

$3 = C$

$$\boxed{y = \frac{\ln x}{x} - \frac{1}{x} + \frac{3}{x^2}}$$

Example. Solve the differential equation $xy' - 2y = x^2$.

$y' - \frac{2}{x}y = x \qquad P(x) = -\frac{2}{x} \Rightarrow I(x) = e^{\int -\frac{2}{x}\,dx}$

$= e^{-2\ln|x|}$

$= e^{\ln|x|^{-2}}$

$= \frac{1}{x^2}$

$\frac{1}{x^2}y' - \frac{2}{x^3}y = \frac{1}{x}$

$\frac{d}{dx}\left(\frac{1}{x^2}\cdot y\right) = \frac{1}{x}$

$\frac{1}{x^2}\cdot y = \int \frac{1}{x}\,dx$

$\frac{1}{x^2}\cdot y = \ln|x| + C$

$\boxed{y = x^2\ln|x| + x^2\cdot C}$

Example. Solve the initial-value problem $t\dfrac{\mathrm{d}u}{\mathrm{d}t} = t^2 + 3u$, $t > 0$, $u(2) = 4$.

$\frac{du}{dt} + P(t)\cdot u = Q(t)$ $\quad t\frac{du}{dt} - 3u = t^2$

$\frac{du}{dt} - \frac{3u}{t} = t$

$P(t) = -\frac{3}{t} \Rightarrow I(t) = e^{\int -\frac{3}{t}dt} = e^{-3\ln|t|} = e^{\ln t^{-3}} = \frac{1}{t^3}$

$\frac{1}{t^3}\frac{du}{dt} - \frac{3}{t^4}\cdot u = \frac{1}{t^2}$

$\frac{d}{dt}\left(\frac{1}{t^3}\cdot u\right) = \frac{1}{t^2}$

$\frac{1}{t^3}\cdot u = \int t^{-2}dt$

$\frac{1}{t^3}\cdot u = -\frac{1}{t} + C$ ←

$u = -t^2 + C\cdot t^3$

$\boxed{u = -t^2 + t^3}$

$t=2, u=4$

$\frac{1}{8}\cdot 4 = -\frac{1}{2} + C$

$\frac{1}{2} + \frac{1}{2} = C$

$1 = C$

Homework 9.5 Solutions: Linear Equations

Prof. Leyda Almodóvar

Name: _____

1. Determine whether the differential equation is linear.

   (a) $y' - x = y \tan x$

   The equation is linear if it can be written in the form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

   Note that $y' - x = y \tan x$ can be rewritten as $y' - y \tan x = x$ where $P(x) = -\tan x$ and $Q(x) = x$. Hence, the differential equation is linear.

   (b) $\frac{dR}{dt} + t \cos R = e^{-t}$

   The equation is linear if it can be written in the form $\frac{dR}{dt} + P(t) \cdot R = Q(t)$

   Notice $\frac{dR}{dt} + t \cos R = e^{-t}$ cannot be rewritten in the standard form. Hence, it is not a linear differential equation.

2. Solve the differential equation.

   (a) $y' - y = e^x$.

   $P(x) = -1$ and $Q(x) = e^x$. Then the integrating factor is $I(x) = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$

   Multiplying both sides of the equation by e^{-x}:

   $e^{-x}y' - e^{-x}y = e^{-x} \cdot e^x$

   $(e^{-x}y)' = 1$

   $e^{-x}y = \int 1\,dx$

   $e^{-x}y = x + C$

   $\boxed{y = xe^x + Ce^x}$

(b) $4x^3y + x^4y' = \sin^3 x$

First we divide both sides of the equation by x^4 to write it in standard form:

$y' + \frac{4}{x}y = x^{-4}\sin^3 x$

Then, $P(x) = \frac{4}{x}$, $Q(x) = \frac{\sin^3 x}{x^4}$ and the integrating factor is $I(x) = e^{\int \frac{4}{x}\,dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4$

Multiplying both sides of the equation by x^4:
$x^4y' + 4x^3y = \sin^3 x$ Note that this is the original equation. That means that the left-hand side was already the derivative of a product.

$(x^4y)' = \sin^3 x$

$x^4y = \int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx = \int -(1-u^2)\,du$
(let $u = \cos x$, then $du = -\sin x$)

$x^4y = -u + \frac{u^3}{3} + C$

$x^4y = \frac{\cos^3 x}{3} - \cos x + C$

$$\boxed{y = \frac{\cos^3 x}{3x^4} - \frac{\cos x}{x^4} + \frac{C}{x^4}}$$

(c) $2xy' + y = 2\sqrt{x}$

First we divide both sides of the equation by $2x$ to write it in standard form:

$y' + \frac{1}{2x}y = x^{-\frac{1}{2}}$

Then, $P(x) = \frac{1}{2x}$, $Q(x) = x^{-\frac{1}{2}}$, and the integrating factor is $I(x) = e^{\int \frac{1}{2x}\,dx} = e^{\frac{1}{2}\int \frac{1}{x}\,dx} = e^{\frac{1}{2}\ln|x|} = x^{\frac{1}{2}}$

Multiplying both sides of the equation by $\sqrt{x}$:

$\sqrt{x}y' + \frac{\sqrt{x}}{2x}y = 1$

$(\sqrt{x}y)' = 1$

$\sqrt{x}y = \int 1\,dx$

$\sqrt{x}y = x + C$

$$\boxed{y = \sqrt{x} + \frac{C}{\sqrt{x}}}$$

(d) $t \ln t \frac{dr}{dt} + r = te^t$

First we divide both sides by $t \ln t$ to write the equation in standard form:

$\frac{dr}{dt} + \frac{r}{t \ln t} = \frac{e^t}{\ln t}$

Then, $P(t) = \frac{1}{t \ln t}$, $Q(t) = \frac{e^t}{\ln t}$, and the integrating factor is $I(t) = e^{\int \frac{1}{t \ln t}\, dt} = e^{\int \frac{1}{u}\, du} = e^{\ln |u|} = |u| = |\ln t|$

Multiplying both sides by $\ln t$:

$\ln t \frac{dr}{dt} + \frac{r}{t} = e^t$

$(r)' = e^t$

$r \ln t = \int e^t \, dt$

$r \ln t = e^t + C$

$$\boxed{r = \frac{e^t + C}{\ln t}}$$

3. Solve the initial-value problem.

(a) $t^3 \frac{dy}{dt} + 3t^2 y = \cos t$, $y(\pi) = 0$.

Note that the left-hand side is already $(t^3 y)'$. Otherwise you will realize it after rewriting it in standard form and finding the integration factor.

$(t^3 y)' = \cos t$

$t^3 y = \int \cos t \, dt$

$t^3 y = \sin t + C$

$y = \frac{\sin t + C}{t^3}$

When $t = \pi$, $y = 0$. Hence, $C = \pi^3(0) - \sin(\pi) = 0$

$$\boxed{y = \frac{\sin t}{t^3}}$$

(b) $xy' + y = x \ln x$, $y(1) = 0$.

Note that the left-hand side is already $(xy)'$.

$(xy)' = x \ln x$

$xy = \int x \ln x \, dx$

We apply integration by parts on the right-hand side where $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$, $\mathrm{d}v = x\,\mathrm{d}x$, and $v = \frac{x^2}{x}$

$xy = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,\mathrm{d}x$

$xy = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

$y = \frac{x\ln x}{2} - \frac{x}{4} + \frac{C}{x}$

When $x = 1$, $y = 0$. Then, $(1)(0) - \frac{1^2}{2}\ln(1) + \frac{1^2}{4} = C$. That is, $C = \frac{1}{4}$.

$$\boxed{y = \frac{x\ln x}{2} - \frac{x}{4} + \frac{1}{4x}}$$

(c) $(x^2+1)\frac{dy}{dx} + 3x(y-1) = 0$, $y(0) = 2$.

We divide both sides by $x^2 + 1$ to write the equation in standard form:

$\frac{dy}{dx} + \frac{3x(y-1)}{x^2+1} = 0$

$\frac{dy}{dx} + \frac{3x}{x^2+1}\cdot y = \frac{3x}{x^2+1}$

Then, $P(x) = \frac{3x}{x^2+1}$, $Q(x) = \frac{3x}{x^2+1}$, and the integrating factor is $I(x) = e^{\int \frac{3x}{x^2+1}\,\mathrm{d}x} = e^{\frac{3}{2}\int \frac{1}{u}\,\mathrm{d}u} = e^{\frac{3}{2}\ln|u|} = e^{\frac{3}{2}\ln|x^2+1|} = e^{\ln|x^2+1|^{\frac{3}{2}}} = (x^2+1)^{\frac{3}{2}}$

Multiplying both sides by $(x^2+1)^{\frac{3}{2}}$:
$(x^2+1)^{\frac{3}{2}}\frac{dy}{dx} + 3xy\sqrt{x^2+1} = 3x\sqrt{x^2+1}$

$[(x^2+1)^{\frac{3}{2}}y]' = 3x\sqrt{x^2+1}$

$(x^2+1)^{\frac{3}{2}}y = \int 3x\sqrt{x^2+1}\,\mathrm{d}x$

$(x^2+1)^{\frac{3}{2}}y = \int \frac{3}{2}\sqrt{u}\,\mathrm{d}u$

$(x^2+1)^{\frac{3}{2}}y = u^{\frac{3}{2}} + C$

$(x^2+1)^{\frac{3}{2}}y = (x^2+1)^{\frac{3}{2}} + C$

When $x = 0$, $y = 2$. Then, $C = 1(2) - (1) = 1$

$$\boxed{y = 1 + \frac{1}{(x^2+1)^{\frac{3}{2}}}}$$

10.2 Calculus with Parametric Curves

Tangent with Parametric Equations

Question. Given a curve $y = f(x)$ and a point $x = a$ on the curve, how do we find the equation of the tangent line at $x = a$?

$m_T = f'(a)$ ← slope of tangent line

$y = f(a) + m_T(x - a)$ ← equation of tangent line

Question. If $y = f(x)$ and $x = g(t)$, how could we find the direction of the parametric curve?

$y = f(g(t))$

$y' = f'(g(t)) \cdot g'(t)$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Fact. If $y = f(x)$ and $x = g(t)$, then $\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$ provided that $\frac{dx}{dt} \neq 0$.

Note. Notice $\frac{dy}{dx}$ is a function of t, not x. Similarly, $\boxed{\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}}$ provided that $\frac{dy}{dt} \neq 0$.

There could be multiple

Example. Find the tangent line(s) to the parametric curve given by $x = t^5 - 4t^3$, $y = t^2$ at (0,4). *x=y

$\frac{dy}{dt} = 2t$

$\frac{dx}{dt} = 5t^4 - 12t^2$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$

Need corresponding t values for (0,4)

$0 = t^5 - 4t^3$ and $4 = t^2$

$0 = t^3(t^2 - 4)$ $t = 2, t = -2$

~~$t = 0$~~, $t = 2, t = -2$

* We only keep t values that satisfy both equations

$t = 2$: $m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{2}{5(2)^3 - 12(2)} = \frac{2}{40 - 24} = \frac{2}{16} = \frac{1}{8}$

$y = f(0) + m_T(x - 0)$

$= 4 + \frac{1}{8} \cdot x$

$\Rightarrow \boxed{y = \frac{1}{8}x + 4}$

$t = -2$: $m = \left.\frac{dy}{dx}\right|_{t=-2} = \frac{2}{5(-2)^3 - 12(-2)} = \frac{2}{-40 + 24} = -\frac{1}{8}$

$y = 4 - \frac{1}{8}(x - 0)$

$\boxed{y = -\frac{1}{8}x + 4}$

Note: If the point is given in terms of t, substitute t in the parametric equations to find the corresponding (x, y) coordinate.

Question. When do we obtain a horizontal tangent? When do we obtain a vertical tangent?

When $\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dt} = 0}$ $\left(\frac{dx}{dt} \neq 0\right)$

$\boxed{\frac{dx}{dt} = 0}$ $\left(\frac{dy}{dt} \neq 0\right)$

Example. Determine the (x, y) coordinates of the points where the parametric equations $x = t^3 - 3t$, $y = 3t^2 - 9$ will have horizontal or vertical tangents.

Horizontal: $\frac{dy}{dt} = 6t = 0 \Rightarrow t = 0$

$x = 0^3 - 3(0) = 0$
$y = 3(0)^2 - 9 = -9$ $\Rightarrow \boxed{(0, -9)}$

Vertical: $\frac{dx}{dt} = 3t^2 - 3 = 0 \Rightarrow 3(t^2 - 1) = 0$
$3(t-1)(t+1) = 0$
$t = 1, t = -1$

$t = 1$:
$x = (1)^3 - 3(1) = -2$
$y = 3(1)^2 - 9 = -6$ $\Rightarrow \boxed{(-2, -6)}$

$t = -1$: $x = (-1)^3 - 3(-1) = 2$
$y = 3(-1)^2 - 9 = -6$ $\Rightarrow \boxed{(2, -6)}$

Fact. The second derivative of y with respect to x is given by $\boxed{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = \dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}}$

Note. $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \dfrac{\frac{\mathrm{d}^2y}{\mathrm{d}t^2}}{\frac{\mathrm{d}^2x}{\mathrm{d}t^2}}$

Example. Find the second derivative for the following set of parametric equations $x = t^5 - 4t^3$, $y = t^2$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t} = 2(5t^3 - 12t)^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-2(5t^3 - 12t)^{-2} \cdot (15t^2 - 12)}{5t^4 - 12t^2}$$

$$= \boxed{\frac{-2(15t^2 - 12)}{(5t^4 - 12t^2)(5t^3 - 12t)^2}} = \frac{-30t^2 + 24}{t(5t^3 - 12t)^3}$$

Question. When is a curve concave upward? When is it concave downward?

When $\frac{d^2y}{dx^2} > 0$ When $\frac{d^2y}{dx^2} < 0$

Example. Determine the values of t for which the parametric curve given by $x = 1 - t^2$, $y = t^7 + t^5$ is concave upward and concave down.

$\frac{dx}{dt} = -2t$, $\frac{dy}{dt} = 7t^6 + 5t^4$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{7t^6+5t^4}{-2t} = -\frac{1}{2}(7t^5+5t^3)$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{2}(35t^4+15t^2)}{-2t} = \frac{1}{4}(35t^3+15t)$

$\frac{d^2y}{dx^2} = 0$ when $\frac{1}{4}(35t^3+15t) = 0$

$\frac{1}{4} \neq 0 \Rightarrow 5t(7t^2+3) = 0$

$5t = 0$ or $7t^2+3 = 0$

$t = 0$ $7t^2 = -3$ ↙ imaginary

− 0 +

$t = 0$

The curve is C.D. when $t < 0$ and C.U. when $t > 0$.

Arc Length with Parametric Equations

Fact. The *arc length* of the parametric curve given by $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ is given by

$$\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example. Determine the length of the parametric curve given by the ~~following parametric curve given by~~ the following parametric equations $x = 3\sin t$, $y = 3\cos t$, $0 \le t \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\cos^2 t + 9\sin^2 t}\, dt = \int_0^{2\pi} \sqrt{9(\underbrace{\cos^2 t + \sin^2 t}_{=1})}\, dt$$

$$= \int_0^{2\pi} \sqrt{9}\, dt = 3t\Big]_0^{2\pi} = \boxed{6\pi}$$

Question. What should be the equation for surface area of a parametric curve?

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{(rotation about x-axis)}$$

$$S = \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{(rotation about y-axis)}$$

Example. Determine the surface area of the solid obtained by rotating $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$ about the x-axis.

$$\frac{dx}{dt} = 3\cos^2 t \cdot (-\sin t)$$

$$\frac{dy}{dt} = 3\sin^2 t \cdot (\cos t)$$

$$S = \int_0^{\pi/2} 2\pi y \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3 t \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3 t \sqrt{9\cos^2 t \sin^2 t (\underbrace{\cos^2 t + \sin^2 t}_{=1})}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3 t \sqrt{9\cos^2 t \sin^2 t}\, dt$$

$$= 2\pi \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t\, dt$$

$$= 6\pi \int_0^{\pi/2} \cos t \sin^4 t\, dt$$

$u = \sin t$, $du = \cos t\, dt$

$t = 0$, $u = \sin 0 = 0$

$t = \pi/2$, $u = \sin \pi/2 = 1$

$$= 6\pi \int_0^1 u^4\, du = 6\pi \cdot \left[\frac{u^5}{5}\right]_0^1 = \boxed{\frac{6\pi}{5}}$$

Homework 10.2 Solutions: Calculus with Parametric Curves

Prof. Leyda Almodóvar

Name: _____

1. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

   (a) $x = t^3 + 1,\ y = t^4 + t;\ t = -1$

   $\frac{dy}{dt} = 4t^3 + 1,\ \frac{dx}{dt} = 3t^2$, and $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t^3+1}{3t^2}$

   When $t = -1$, $(x, y) = (0, 0)$ and $\frac{dy}{dx} = \frac{-3}{3} = -1$, so an equation of the tangent to the curve at the point corresponding to $t = -1$ is $\boxed{y = -x}$

   (b) $x = t\cos t,\ y = t\sin t;\ t = \pi$ $\frac{dy}{dt} = t\cos t + \sin t,\ \frac{dx}{dt} = t(-\sin t) + \cos t$, and $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t\cos t+\sin t}{t(-\sin t)+\cos t}$

   When $t = \pi$, $(x, y) = (-\pi, 0)$ and $\frac{dy}{dx} = \frac{-\pi}{-1} = \pi$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $\boxed{y = \pi x + \pi^2}$

2. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

   (a) $x = t^2 + 1,\ y = t^2 + t$

   $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$

   $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{-1}{(2t^2)}}{2t} = -\frac{1}{4t^3}$

   The curve is concave upward when $\frac{d^2y}{dx^2} > 0$, that is, when $t < 0$.

(b) $x = e^t$, $y = te^{-t}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-te^{-t}+e^{-t}}{e^t} = \frac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t)$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{e^{-2t}(-1)+(1-t)(-2e^{-2t})}{e^t} = \frac{e^{-2t}(-1-2+2t)}{e^t} = e^{-3t}(2t-3)$

The curve is concave upward when $\frac{d^2y}{dx^2} > 0$, that is, when $t > \frac{3}{2}$.

3. Find the points on the curve $x = t^3 - 3t$, $y = t^2 - 3$ where the tangent is horizontal or vertical.

$\frac{dy}{dt} = 2t$, so $\frac{dy}{dt} = 0$ when $t = 0$, which yields $(x, y) = (0, -3)$.
$\frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\frac{dx}{dt} = 0$ when $t = -1$ or $t = 1$, which yields $(x, y) = (2, -2)$ or $(-2, -2)$.

The curve has a horizontal tangent at $(0, -3)$ and vertical tangents at $(2, -2)$ and $(-2, -2)$.

4. Find the exact length of the curve $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.

$\frac{dx}{dt} = 6t$ and $\frac{dy}{dt} = 6t^2$, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2 + 36t^4$

$$L = \int_0^1 \sqrt{36t^2 + 36t^4}\,dt = \int_0^1 6t\sqrt{1+t^2}\,dt = 6\int_1^2 \sqrt{u}\left(\tfrac{1}{2}\,du\right) = 3\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_1^2 =$$
$$2(2^{\frac{3}{2}} - 1) = \boxed{2(2\sqrt{2}-1)}$$

5. Find the exact area of the surface obtained by rotating the curve $x = t^3$, $y = t^2$, $0 \le t \le 1$ about the x-axis.

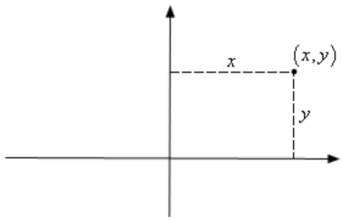
$\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 2t$, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^4 + 4t^2$

$$S = \int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi t^2\sqrt{9t^4+4t^2}\,dt = 2\pi\int_0^1 t^2\sqrt{t^2(9t^2+4)}\,dt$$
$$= 2\pi\int_4^{13}\left(\tfrac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}\,du\right) = \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{8}{3}u^{\frac{3}{2}}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{\frac{5}{2}} - 20u^{\frac{3}{2}}\right]_4^{13}$$

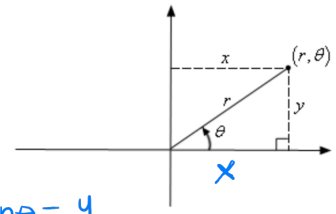
$$= \frac{2\pi}{1215}[(3 \cdot 13^2\sqrt{13} - 20 \cdot 13\sqrt{13}) - (3 \cdot 32 - 20 \cdot 8)] = \boxed{\frac{2\pi}{1215}(247\sqrt{13} + 64)}$$

10.3 Polar Coordinates

In the Cartesian coordinate system a point is given by the (x, y) coordinates, where we start at the origin and then move x units horizontally followed by y units vertically.



Another way to define a point in two-dimensional space:



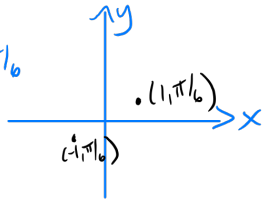
measure distance from origin to the point and the angle between the positive x-axis and the line connecting origin to the point

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

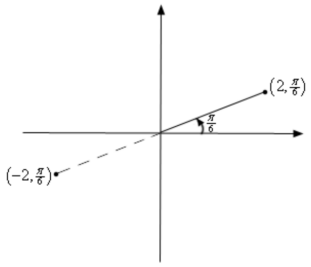
$\sin\theta = \frac{y}{r} \Rightarrow y = r\cos\theta$

Fact. Coordinates in this form are called **polar coordinates**. Note that we also allow r to be negative.

- If $r > 0$, the point (r, θ) will be in the same quadrant as θ. $(1, \pi/6)$
- If $r < 0$, the point (r, θ) will end up in the quadrant exactly opposite θ. $(-1, \pi/6)$
- $(-1, \frac{\pi}{6})$ describes the same point as the coordinates $(1, \frac{7\pi}{6})$ do.
- In general, $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.
- In Cartesian coordinates there is exactly one set of coordinates for any given point. In polar coordinates there is an infinite number of coordinates for a given point.
- In polar coordinates the origin is often called the **pole**.



represented by $(0, \theta)$



$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

Note. To convert from polar to cartesian: $\boxed{x = r\cos\theta, y = r\sin\theta}$.

$\tan\theta = \frac{y}{x}$

To convert from cartesian to polar: $\boxed{r = \sqrt{x^2+y^2}}$ (assume $r > 0$) and $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and is given by $\theta + \pi$.

Example. Convert $(-4, \frac{2\pi}{3})$ into Cartesian coordinates.

Given $r=-4$ and $\theta = \frac{2\pi}{3}$

$x = r\cos\theta = -4\cos\frac{2\pi}{3} = -4\cdot\left(-\frac{1}{2}\right) = 2$

$y = r\sin\theta = -4\sin\frac{2\pi}{3} = -4\cdot\frac{\sqrt{3}}{2} = -2\sqrt{3}$

$\Rightarrow \boxed{(2, -2\sqrt{3})}$

Example. Convert $(-1,-1)$ into polar coordinates.

Given $x=-1$ and $y=-1$

$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \pi/4$

Note $(\sqrt{2}, \pi/4)$ lies in the first quadrant while $(-1,-1)$ lies in the third quadrant.

Correct point is $\boxed{(\sqrt{2}, 5\pi/4)}$ or $\boxed{(-\sqrt{2}, \pi/4)}$

Example. Convert each of the following into an equation in the given coordinate system.

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.

$2r\cos\theta - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$

$\boxed{2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta}$

(b) Convert $r = -8\cos\theta$ into Cartesian coordinates.

$\sqrt{x^2+y^2} = -8\cdot\frac{x}{\sqrt{x^2+y^2}}$

$x = r\cos\theta$

$\frac{x}{r} = \cos\theta$

$x^2+y^2 = -8x$

$x^2+8x+y^2 = 0$

$x^2+8x+\left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + y^2 = 0$

$(x+4)^2 + y^2 = 16$ ← equation of a circle of radius 4 centered at $(-4,0)$

Common Polar Coordinate Graphs

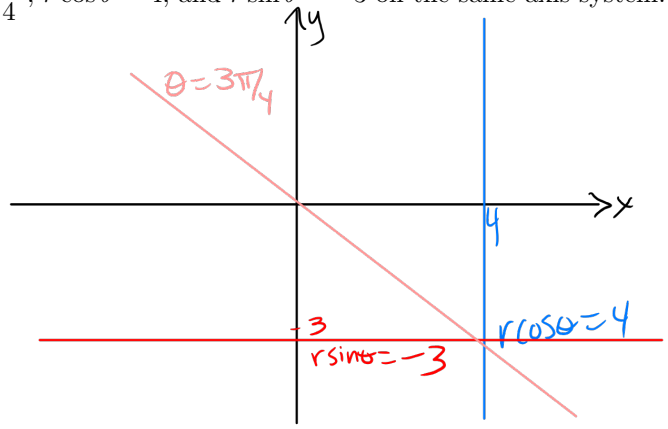
Lines

line with $m = \tan\beta$

1. $\theta = \beta$ implies $\tan^{-1}\left(\frac{y}{x}\right) = \beta$, hence $\frac{y}{x} = \tan\beta$, which is a number. Thus, $y = (\tan\beta)x$ is a line that goes through the origin and makes an angle of β with the positive x-axis (or slope equal to $\tan\beta$).
2. $r\cos\theta = a$ implies $x = a$. This is a vertical line.
3. $r\sin\theta = b$ implies $y = b$. This is a horizontal line.

Example. Graph $\theta = \frac{3\pi}{4}$, $r\cos\theta = 4$, and $r\sin\theta = -3$ on the same axis system.

$x=4$ $\quad y=-3$



Circles

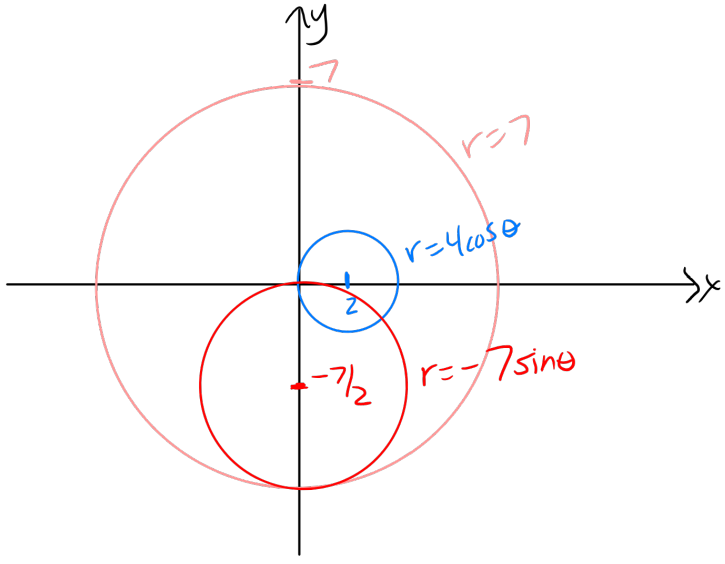
1. $r = a$. This equation is saying that no matter what angle we've got, the distance from the origin must be a. That's precisely the definition of a circle of radius a centered at the origin.
2. $r = 2a\cos\theta$ is a circle of radius $|a|$ and center $(a, 0)$. In example $a=-4$
3. $r = 2b\sin\theta$ is a circle of radius $|b|$ and center $(0, b)$.
4. $r = 2a\cos\theta + 2b\sin\theta$ is a combination of 2) and 3) and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2 + b^2}$ and center (a, b).

Example. Graph $r = 7$, $r = 4\cos\theta$, and $r = -7\sin\theta$ on the same axis system.

$r=7$: circle of radius 7 centered at the pole

$r=4\cos\theta$: circle of radius 2 centered at $(2,0)$

$r=-7\sin\theta$: circle of radius $\left|-\frac{7}{2}\right|$ centered at $(0, -7/2)$



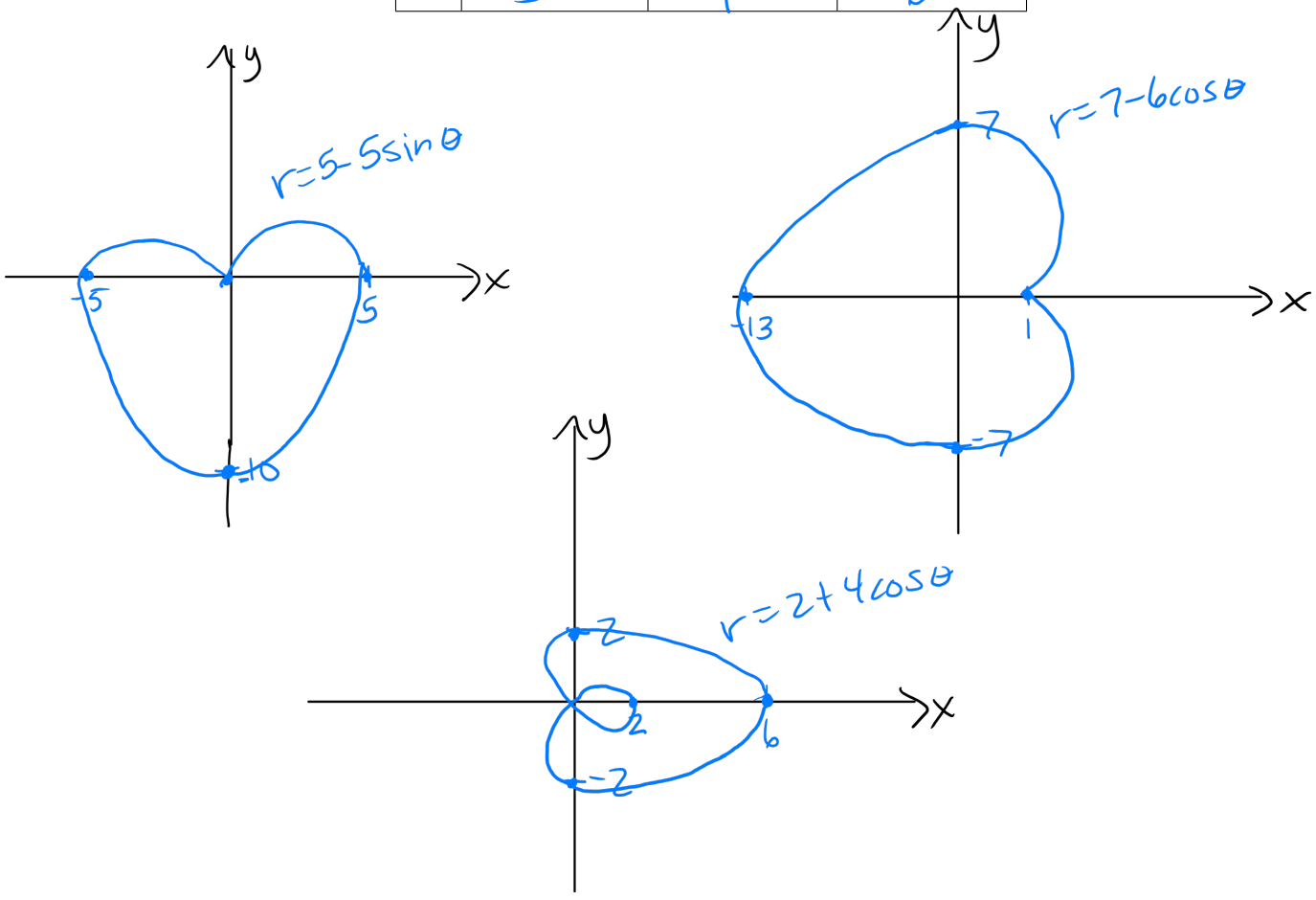
Cardioids and Limaçons (French word for snails)

1. **Cardioids:** $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$. These have a graph that is vaguely heart shaped and always contains the origin.
2. **Limaçons with an inner loops:** $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a < b$. These have an inner loop and will always contain the origin.
3. **Limaçons without an inner loops:** $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a > b$. These do not have an inner loop and do not contain the origin.

Example. Graph $r = 5 - 5\sin\theta$, $r = 7 - 6\cos\theta$, and $r = 2 + 4\cos\theta$.

θ	$r = 5 - 5\sin\theta$ (Cardioid)	$r = 7 - 6\cos\theta$ (Limaçon no loop)	$r = 2 + 4\cos\theta$ (Limaçon with loop)
0	5	1	6
$\frac{\pi}{2}$	0	7	2
π	5	13	−2
$\frac{3\pi}{2}$	10	7	2
2π	5	1	6

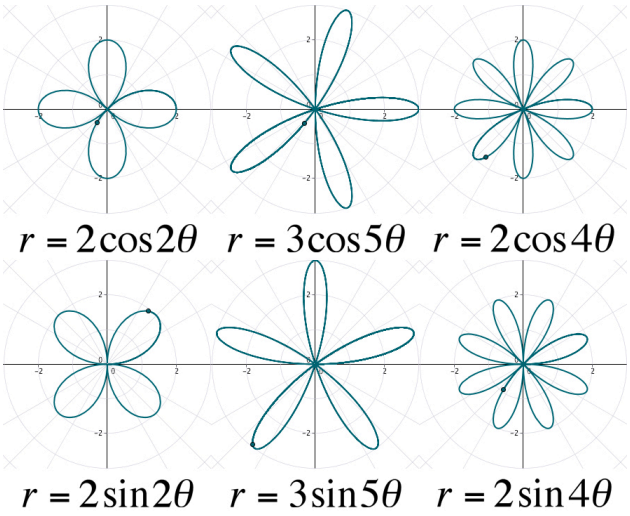
Loop!



Rose curves

A rose curve is a graph that is produced from a polar equation in the form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ where $a \neq 0$, $n > 1$ is an integer.

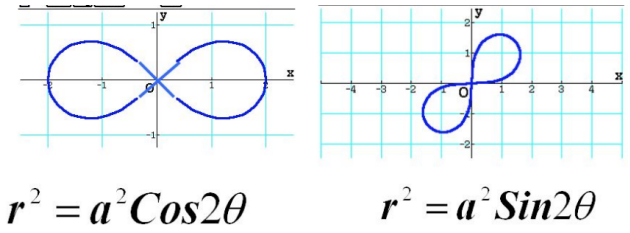
They are called rose curves because the loops that are formed resemble petals. The number of petals that are present will depend on the value of n. The value of a will determine the length of the petals.



Lemniscates

$r^2 = a^2\sin(2\theta)$ or $r^2 = a^2\cos(2\theta)$, where $a \neq 0$. They have the shape of a figure 8.

A lemniscate containing the sine function will be symmetric to the pole while a lemniscate containing the cosine function will be symmetric to the polar axis, to $\theta = \dfrac{\pi}{2}$, and the pole.



Tangents with Polar Coordinates

Product rule

Note. If $r = f(\theta)$, $x = r\cos\theta$, $y = r\sin\theta$ then $\boxed{\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta}$ $\boxed{\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta}$ and

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Example. Find the slope of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/6}$ $\quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\cdot\sin\theta + r\cos\theta = 8\cos\theta\cdot\sin\theta + (3+8\sin\theta)\cos\theta$

$= 16\cos\theta\sin\theta + 3\cos\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 8\cos\theta\cdot\cos\theta - (3+8\sin\theta)\sin\theta$

$= 8\cos^2\theta - 3\sin\theta - 8\sin^2\theta$

Recall: $\cos\pi/6 = \frac{\sqrt{3}}{2}$, $\sin\pi/6 = 1/2$

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{16\cos\pi/6\sin\pi/6 + 3\cos\pi/6}{8\cos^2\pi/6 - 3\sin\pi/6 - 8\sin^2\pi/6}$$

$$= \frac{16\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + 3\cdot\frac{\sqrt{3}}{2}}{8\cdot\left(\frac{\sqrt{3}}{2}\right)^2 - 3\cdot\frac{1}{2} - 8\left(\frac{1}{2}\right)^2} = \frac{4\sqrt{3} + \frac{3}{2}\cdot\sqrt{3}}{8\cdot\frac{3}{4} - \frac{3}{2} - 2} = \frac{4\sqrt{3} + \frac{3}{2}\sqrt{3}}{4 - \frac{3}{2}}\cdot\frac{2}{2}$$

$$= \frac{8\sqrt{3} + 3\sqrt{3}}{8-3} = \boxed{\frac{11\sqrt{3}}{5}}$$

Question. When is the tangent line horizontal? When is it vertical?

$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{d\theta} = 0 \quad \left(\frac{dx}{d\theta} \neq 0\right)$

when $\frac{dx}{d\theta} = 0 \quad \left(\frac{dy}{d\theta} \neq 0\right)$

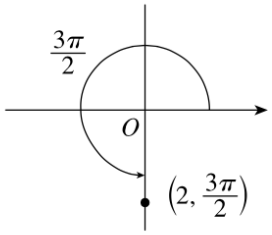
Homework 10.3 Solutions: Polar Coordinates

Prof. Leyda Almodóvar

Name: _____

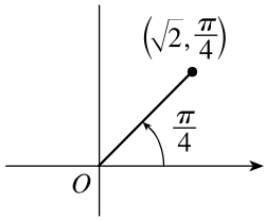
1. Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

   (a) $(2, \frac{3\pi}{2})$



   $x = 2\cos\frac{3\pi}{2} = 2(0) = 0$ and $y = 2\sin\frac{3\pi}{2} = 2(-1) = -2$ give us the Cartesian coordinates $(0, -2)$.

   (b) $(\sqrt{2}, \frac{\pi}{4})$



   $x = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$ and $y = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$ give us the Cartesian coordinates $(1, 1)$.

2. The Cartesian coordinates of a point are given.

   (a) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \le \theta < 2\pi$.
   (b) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \le \theta < 2\pi$.

   (a) $(-4, 4)$

   $r = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ and $\tan\theta = \frac{4}{-4} = -1$, which implies $\theta = -\frac{\pi}{4} + n\pi$.

   Since $(-4, 4)$ is in the second quadrant, the polar coordinates are (i) $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ and (ii) $\left(-4\sqrt{2}, \frac{7\pi}{4}\right)$

   (b) $(3, 3\sqrt{3})$

   $r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$ and $\tan\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}$, which implies $\theta = \frac{\pi}{3} + n\pi$.

   Since $(3, 3\sqrt{3})$ is in the first quadrant, the polar coordinates are (i) $\left(6, \frac{\pi}{3}\right)$ and (ii) $\left(-6, \frac{4\pi}{3}\right)$

3. Identify the curve by finding a Cartesian equation for the curve.

   (a) $r^2 = 5$

   $r^2 = 5$ implies $x^2 + y^2 = 5$ is a circle of radius $\sqrt{5}$ centered at the origin.

   (b) $r = 5\cos\theta$

   $r = 5\cos\theta$ implies $r^2 = 5r\cos\theta$. Hence, $x^2 + y^2 = 5x$. Completing the square we obtain $\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$, a circle of radius $\frac{5}{2}$ centered at $\left(\frac{5}{2}, 0\right)$.

   [Note: The first two equations are actually equivalent since $r^2 = 5r\cos\theta$ implies $r(r - 5\cos\theta) = 0$, $r = 0$ or $r = 5\cos\theta$. But $r = 5\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 5\cos\theta$ is equivalent to the compound condition $r = 0$ or $r = 5\cos\theta$.]

4. Find a polar equation for the curve represented by the given Cartesian equation.

   (a) $y = 2$

   $y = 2$ implies $r\sin\theta = 2$, which implies $\boxed{r = 2\csc\theta}$

   (b) $y = 1 + 3x$

   $y = 1 + 3x$ implies $r\sin\theta = 1 + 3r\cos\theta$. Hence, $r(\sin\theta - 3\cos\theta) = 1$ and $\boxed{r = \dfrac{1}{\sin\theta - 3\cos\theta}}$

5. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

   (a) $r = 2\cos\theta$, $\theta = \frac{\pi}{3}$

   $x = r\cos\theta = 2\cos^2\theta$, $y = r\sin\theta = 2\sin\theta\cos\theta = \sin 2\theta$

   $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\cos 2\theta}{2\cdot 2\cos\theta(-\sin\theta)} = \frac{\cos 2\theta}{-\sin 2\theta} = -\cot 2\theta$

   When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = -\cot\left(2\cdot\frac{\pi}{3}\right) = \cot\frac{\pi}{3} = \boxed{\frac{1}{\sqrt{3}}}$

   (b) $r = \frac{1}{\theta}$, $\theta = \pi$

   $x = r\cos\theta = \frac{\cos\theta}{\theta}$, $y = r\sin\theta = \frac{\sin\theta}{\theta}$

   $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta\left(-\frac{1}{\theta^2}\right)+\left(\frac{1}{\theta}\right)\cos\theta}{\cos\theta\left(-\frac{1}{\theta^2}\right)-\left(\frac{1}{\theta}\right)\sin\theta}\cdot\frac{\theta^2}{\theta^2} = \frac{-\sin\theta+\theta\cos\theta}{-\cos\theta-\theta\sin\theta}$

   When $\theta = \pi$, $\frac{dy}{dx} = -\frac{-\sin\pi+\pi\cos\pi}{-\cos\pi-\pi\sin\pi} = \frac{-\pi}{1} = \boxed{-\pi}$

6. Find the points on the given curve where the tangent line is horizontal or vertical.

   (a) $r = 3\cos\theta$

   $x = r\cos\theta = 3\cos\theta\cos\theta$, $y = r\sin\theta = 3\cos\theta\sin\theta$

   $\frac{dy}{d\theta} = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0$, when $2\theta = \frac{\pi}{2}$ or $2\theta = \frac{3\pi}{2}$. Hence, $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$

   The tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$

   $\frac{dx}{d\theta} = -6\sin\theta\cos\theta = -3\sin 2\theta = 0$ when $2\theta = 0$ or $2\theta = \pi$. Hence, $\theta = 0$ or $\theta = \frac{\pi}{2}$.

   The tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$

   (b) $r = 1 + \cos\theta$

   $x = r\cos\theta = \cos\theta(1+\cos\theta)$, $y = r\sin\theta = \sin\theta(1+\cos\theta)$

   $\frac{dy}{d\theta} = (1+\cos\theta)\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$ when $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$, which implies $\theta = \frac{\pi}{3}$, $\theta = \pi$, or $\theta = \frac{5\pi}{3}$.

   The tangent is horizontal at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$

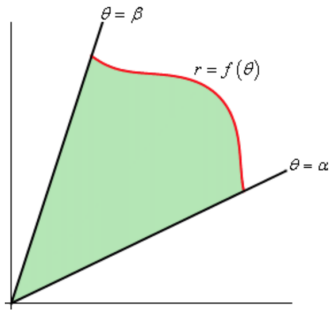
$\frac{dx}{d\theta} = (-1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta(1 + 2\cos\theta) = 0$ when $\sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$.
Hence, $\theta = 0$, $\theta = \pi$, $\theta = \frac{2\pi}{3}$, or $\theta = \frac{4\pi}{3}$.

The tangent is vertical at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

[Note: The tangent is horizontal, not vertical when $\theta = \pi$]

10.4 Areas and Length in Polar Coordinates

In this section we are going to find areas enclosed by polar curves.



Fact. The area of the region enclosed by the curves $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is given by $\boxed{A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta}$

Example. Determine the area of the inner loop of $2 + 4\cos\theta$.

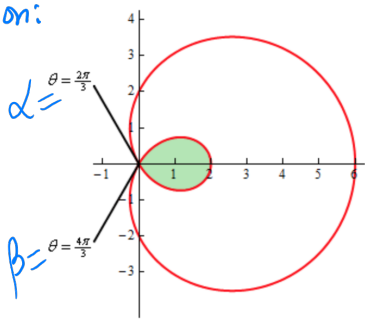
To find limits of integration:

$0 = 2 + 4\cos\theta$

$-2 = 4\cos\theta$

$-1/2 = \cos\theta$

$\theta = 2\pi/3, 4\pi/3$



Recall: $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin 2\pi/3 = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$

$\sin 4\pi/3 = -\frac{\sqrt{3}}{2}$

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2 + 4\cos\theta)^2 \, d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(4 + 16\cos\theta + 16\cos^2\theta) \, d\theta$$

$$= \int_{2\pi/3}^{4\pi/3} 2 + 8\cos\theta + 8 \cdot \frac{1}{2}(1 + \cos 2\theta) \, d\theta = \int_{2\pi/3}^{4\pi/3} 6 + 8\cos\theta + 4\cos 2\theta \, d\theta$$

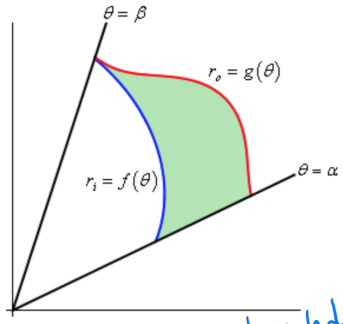
$$= 6\theta + 8\sin\theta + \frac{4 \cdot \sin(2\theta)}{2} \Big]_{2\pi/3}^{4\pi/3} = 6 \cdot \frac{4\pi}{3} + 8\sin 4\pi/3 + 2 \cdot \sin 8\pi/3$$

$$- \left(6 \cdot 2\pi/3 + 8 \cdot \sin 2\pi/3 + 2 \cdot \sin 4\pi/3\right) = 8\pi + 8 \cdot \left(-\frac{\sqrt{3}}{2}\right) + 2 \cdot \left(\frac{\sqrt{3}}{2}\right)$$

$$- 4\pi - 8 \cdot \frac{\sqrt{3}}{2} - 2 \cdot \left(\frac{\sqrt{3}}{2}\right) = \boxed{4\pi - 6\sqrt{3}}$$

Fact. The area of the region enclosed by the curves $r_\theta = g(\theta)$, $r_i = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is given by

$$A = \int_\alpha^\beta \frac{1}{2}\left(r_\theta^2 - r_i^2\right) \mathrm{d}\theta$$

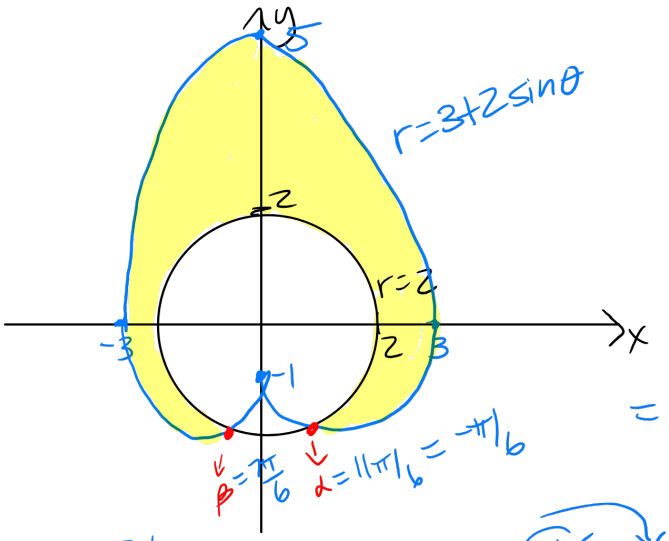


the region bounded by

Limaçon without a loop

θ	$r = 3 + 2\sin\theta$
0	3
$\pi/2$	5
π	3
$3\pi/2$	1

Example. Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.



$r = 3+2\sin\theta$

$2 = 3+2\sin\theta \Rightarrow -1 = 2\sin\theta \Rightarrow \sin\theta = -\frac{1}{2} \rightarrow \theta = 7\pi/6$ or $\theta = 11\pi/6$

$$A = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[(3+2\sin\theta)^2 - 2^2\right] d\theta$$

$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[9 + 12\sin\theta + 4\sin^2\theta - 4\right] d\theta$$

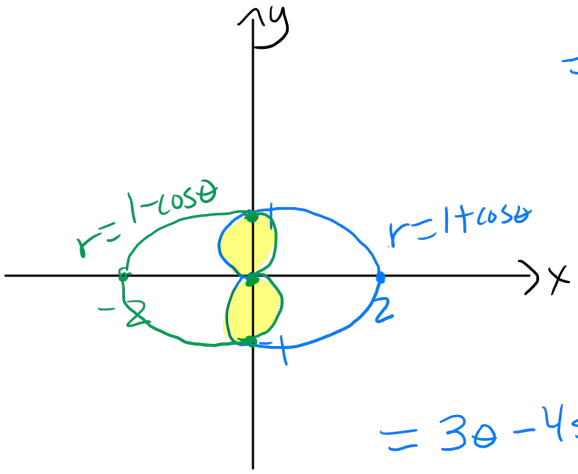
$$= \int_{-\pi/6}^{7\pi/6} \frac{1}{2}\left[5 + 12\sin\theta + 4\cdot\frac{1}{2}(1-\cos 2\theta)\right] d\theta = \frac{1}{2}\int_{-\pi/6}^{7\pi/6} 7 + 12\sin\theta - 2\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[7\theta - 12\cos\theta - \frac{2\sin(2\theta)}{2}\right]_{-\pi/6}^{7\pi/6} = \frac{1}{2}\left[7\cdot\frac{7\pi}{6} - 12\cos\frac{7\pi}{6} - \sin\frac{14\pi}{6}\right.$$

$$\left. - \left(-\frac{7\pi}{6} - 12\cos\left(-\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{3}\right)\right)\right] = \cdots = \boxed{\frac{14\pi}{3} + \frac{11\sqrt{3}}{2}}$$

Example. Find the area of the region that lies inside both curves $r = 1 + \cos\theta$, $r = 1 - \cos\theta$.

θ	$r=1+\cos\theta$	$r=1-\cos\theta$
0	2	0
$\pi/2$	1	1
π	0	2
$3\pi/2$	1	1



Note: There's symmetry!

$$A = 4\cdot\int_0^{\pi/2}\frac{1}{2}(1-\cos\theta)^2\,d\theta$$

$$= 2\int_0^{\pi/2} 1-2\cos\theta+\cos^2\theta\,d\theta$$

$$= 2\int_0^{\pi/2} 1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} 3/2-2\cos\theta+\frac{1}{2}\cos 2\theta\,d\theta$$

$$= 2\left[\frac{3}{2}\cdot\theta-2\sin\theta+\frac{1}{2}\cdot\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 3\theta-4\sin\theta+\frac{1}{2}\sin 2\theta\Big]_0^{\pi/2}$$

$$= \frac{3\pi}{2}-4\cdot\sin\pi/2+\frac{1}{2}\sin\pi-\left(3\cdot 0-4\sin 0+\frac{1}{2}\sin 0\right)$$

$$= \boxed{\frac{3\pi}{2}-4}$$

Arc Length

The arc length of the curve given by $r = f(\theta)$, $\alpha \le \theta \le \beta$ is $\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$

Example. Determine the length of $r = \theta$, $0 \le \theta \le 1$.

$\frac{dr}{d\theta} = 1$

$\theta = \tan x$

$d\theta = \sec^2 x\, dx$

if $\theta = 0$, $0 = \tan x \Rightarrow x = 0$

$\theta = 1$, $1 = \tan x \Rightarrow x = \pi/4$

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta = \int_0^{\pi/4} \sqrt{\tan^2 x + 1} \cdot \sec^2 x\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x \cdot \sec^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec^3 x\, dx = \sec x \tan x \Big]_0^{\pi/4} - \int_0^{\pi/4} \sec x \tan^2 x\, dx$$

$u = \sec x \quad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx \quad v = \tan x$

$$= \sec x \tan x \Big]_0^{\pi/4} - \int_0^{\pi/4} \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x \Big]_0^{\pi/4} - \int_0^{\pi/4} \sec^3 x\, dx + \int_0^{\pi/4} \sec x\, dx$$

$$\Rightarrow 2 \cdot \int_0^{\pi/4} \sec^3 x\, dx = \sec \pi/4 \tan \pi/4 - \sec 0 \cdot \tan 0 + \ln|\sec x + \tan x| \Big]_0^{\pi/4}$$

$$= \sqrt{2} \cdot 1 - 0 + \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$= \sqrt{2} + \ln|\sqrt{2} + 1| - \ln 1 = \sqrt{2} + \ln|\sqrt{2} + 1|$$

$$\boxed{L = \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2} + 1)\right)}$$

Homework 10.4: Areas and Length in Polar Coordinates

Prof. Leyda Almodóvar

Name: _____

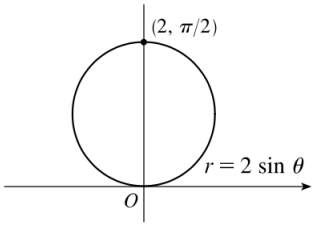
1. Sketch the curve and find the area that it encloses

   (a) $r = 2\sin\theta$

   The area is bounded by $r = 2\sin\theta$ for $\theta = 0$ to $\theta = \pi$.

   $A = \int_0^\pi \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^\pi (2\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi 4\sin^2\theta\,d\theta = 2\int_0^\pi \frac{1}{2}(1-\cos 2\theta)\,d\theta$

   $= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^\pi = \boxed{\pi}$

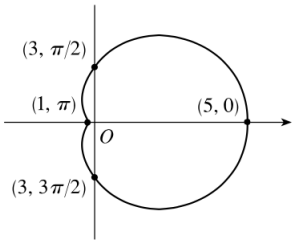


   (b) $r = 3 + 2\cos\theta$

   $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (3+2\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (9+12\cos\theta+4\cos^2\theta)\,d\theta$

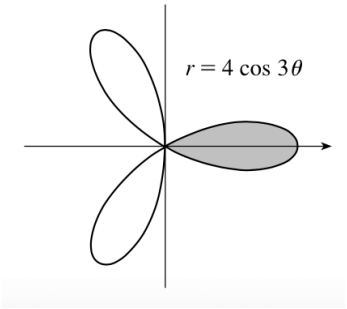
   $= \frac{1}{2}\int_0^{2\pi} [9+12\cos\theta+2(1+\cos 2\theta)]\,d\theta = \frac{1}{2}\int_0^{2\pi} (11+12\cos\theta+2\cos^2\theta)\,d\theta$

   $= \frac{1}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{2\pi} = \frac{1}{2}(22\pi) = \boxed{11\pi}$



2. Find the area of the region enclosed by one loop of the curve.

(a) $r = 4\cos 3\theta$



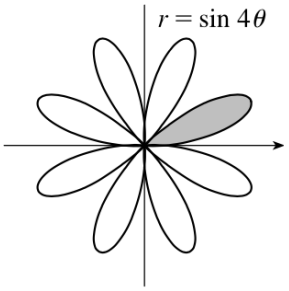
The curve passes through the pole when $r = 0$.

$4\cos 3\theta = 0$ when $\cos 3\theta = 0$. Hence, $3\theta = \frac{\pi}{2} + \pi n$ and $\theta = \frac{\pi}{6} + \frac{\pi}{3}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \frac{\pi}{6}$, so we'll use $-\frac{\pi}{6}$ and $\frac{\pi}{6}$ as our limits of integration.

$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(4\cos 3\theta)^2 \, d\theta = 2\int_0^{\frac{\pi}{6}} \frac{1}{2}(16\cos^2 3\theta) \, d\theta = 16\int_0^{\frac{\pi}{6}} \frac{1}{2}(1 + \cos 6\theta) \, d\theta$

$= 8\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{6}} = 8\left(\frac{\pi}{6}\right) = \boxed{\frac{4\pi}{3}}$

(b) $r = \sin 4\theta$



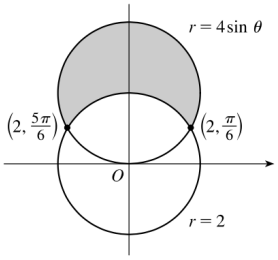
$r = 0$ when $\sin 4\theta = 0$. That is, $4\theta = \pi n$ and $\theta = \frac{\pi}{4}n$.

$A = \int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin 4\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2 4\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 8\theta) \, d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{4}\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi}{16}}$

3. Find the area of the region that lies inside the first curve and outside the second curve.

(a) $r = 4\sin\theta$, $r = 2$



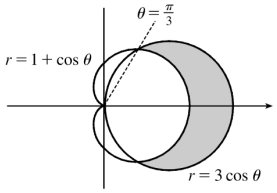
The curves intersect when $4\sin\theta = 2$, that is $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. Then,

$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}[(4\sin\theta)^2 - 2^2] \, d\theta = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(16\sin^2\theta - 4) \, d\theta$ [By symmetry]

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[16\frac{1}{2}(1 - \cos 2\theta) - 4\right] d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4 - 8\cos 2\theta) \, d\theta = \left[4\theta - 4\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= (2\pi - 0) - \left(\frac{2\pi}{3} - 2\sqrt{3}\right) = \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$

(b) $r = 3\cos\theta$, $r = 1 + \cos\theta$



$3\cos\theta = 1 + \cos\theta$ implies that $\cos\theta = \frac{1}{2}$. Hence, $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$.

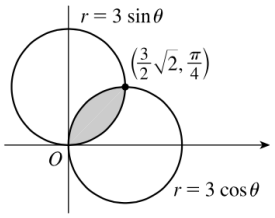
$A = 2\int_0^{\frac{\pi}{3}} \frac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta = \int_0^{\frac{\pi}{3}} (8\cos^2\theta - 2\cos\theta - 1)\, d\theta$

$= \int_0^{\frac{\pi}{3}} [4(1+\cos 2\theta) - 2\cos\theta - 1]\, d\theta = \int_0^{\frac{\pi}{3}} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta$

$= [3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\frac{\pi}{3}} = \pi + \sqrt{3} - \sqrt{3} = \boxed{\pi}$

4. Find the area of the region that lies inside both curves.

(a) $r = 3\sin\theta$, $r = 3\cos\theta$

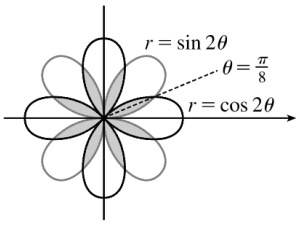


$3\sin\theta = 3\cos\theta$ implies $\tan\theta = 1$. Hence, $\theta = \frac{\pi}{4}$

$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\left[(3\sin\theta)^2\, d\theta = \int_0^{\frac{\pi}{4}} 9\sin^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} 9 \cdot \frac{1}{2}(1 - \cos 2\theta)\, d\theta\right.$

$= \int_0^{\frac{\pi}{4}} \left(\frac{9}{2} - \frac{9}{2}\cos 2\theta\right) d\theta = \left[\frac{9}{2}\theta - \frac{9}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \left(\frac{9\pi}{8} - \frac{9}{4}\right) - (0 - 0) = \boxed{\frac{9\pi}{8} - \frac{9}{4}}$

(b) $r = \sin 2\theta$, $r = \cos 2\theta$



$\sin 2\theta = \cos 2\theta$ implies $\tan 2\theta = 1$. Hence, $2\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{8}$

$A = 8 \cdot 2 \int_0^{\frac{\pi}{8}} \frac{1}{2} \sin^2 2\theta \, d\theta = 8 \int_0^{\frac{\pi}{8}} \frac{1}{2}(1 - \cos 4\theta) \, d\theta = 4\left[\theta - \frac{1}{4} \sin 4\theta\right]_0^{\frac{\pi}{8}}$

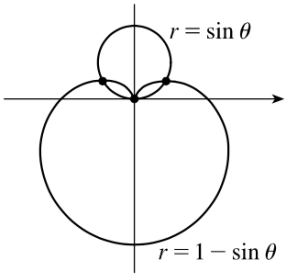
$= 4\left(\frac{\pi}{8} - \frac{1}{4} \cdot 1\right) = \boxed{\frac{\pi}{2} - 1}$

5. Find all points of intersection of the given curves.

(a) $r = \sin\theta$, $r = 1 - \sin\theta$

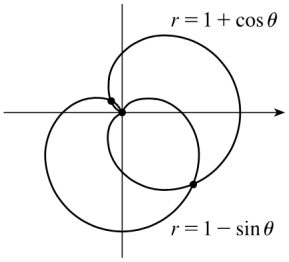
By sketching the curves, we can see that the pole is a point of intersection. Note that the pole has coordinates $(0, \theta)$ for any θ. Additionally,

$\sin\theta = 1 - \sin\theta$ implies $\sin\theta = \frac{1}{2}$. Hence, $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. The other points of intersection are $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.



(b) $r = 1 + \cos\theta$, $r = 1 - \sin\theta$

By sketching the curves, we can see that the pole is a point of intersection. Additionally,



$1 + \cos\theta = 1 - \sin\theta$ implies $\cot\theta = -1$. Hence, $\theta = \frac{3\pi}{4}$ or $\theta = \frac{7\pi}{4}$. The other points of intersection are $\left(1 - \frac{1}{2}\sqrt{2}, \frac{3\pi}{4}\right)$ and $\left(1 + \frac{1}{2}\sqrt{2}, \frac{7\pi}{4}\right)$.

6. Find the exact length of the polar curve $r = 2\cos\theta$, $0 \le \theta \le \pi$.

$L = \int_a^b \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)}\,\mathrm{d}\theta = \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\,\mathrm{d}\theta = \int_0^\pi \sqrt{4(\cos^2\theta + \sin^2\theta)}\,\mathrm{d}\theta$

$= \int_0^\pi \sqrt{4}\,\mathrm{d}\theta = [2\theta]_0^\pi = \boxed{2\pi}$

11 Further Applications of Integration

11.1 Arc Length

In this section we find the length of a given curve.

The Arc Length Formula

If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is $L = \int_a^b \sqrt{1+[f'(x)]^2}\,\mathrm{d}x$.

In Leibniz notation $\boxed{L = \int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}$

If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging x and y

In the formula we get: $\boxed{L = \int_c^d \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y}$

Example. Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.

$y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{1/2}$

$L = \int_1^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$

$u = 1+\frac{9}{4}x \qquad x=1,\ u=\frac{13}{4}$

$du = \frac{9}{4}dx \qquad x=4,\ u=10$

$\frac{4}{9}du = dx$

$= \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big]_{13/4}^{10}$

$= \boxed{\frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]}$

Example. Find the length of the arc of $y = \frac{x^3}{3} + \frac{1}{4x}$ where $1 \le x \le 2$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{3x^2}{3} - \frac{1}{4x^2} = x^2 - \frac{1}{4x^2}$$

$$= \int_1^2 \sqrt{1 + x^4 - 2 \cdot \frac{x^2}{4x^2} + \frac{1}{16x^4}}\, dx$$

$$= \int_1^2 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx = \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}}\, dx = \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$$

$$= \int_1^2 x^2 + \frac{1}{4} \cdot x^{-2}\, dx = \frac{x^3}{3} - \frac{1}{4} \cdot \frac{1}{x} \Big]_1^2 = \frac{8}{3} - \frac{1}{8} - \left(\frac{1}{3} - \frac{1}{4}\right) = \cdots = \boxed{\frac{59}{24}}$$

Example. Find the length of the arc of $x = \frac{y^4}{8} + \frac{1}{4y^2}$ from $y = 1$ to $y = 2$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^2 \sqrt{1 + \left(\frac{y^3}{2} - \frac{1}{2y^3}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{4y^3}{8} - \frac{2}{4y^3} = \frac{y^3}{2} - \frac{1}{2y^3}$$

$$= \int_1^2 \sqrt{1 + \frac{y^6}{4} - \frac{1}{2} + \frac{1}{4y^6}}\, dy$$

$$= \int_1^2 \sqrt{\frac{y^6}{4} + \frac{1}{2} + \frac{1}{4y^6}}\, dy = \int_1^2 \sqrt{\left(\frac{y^3}{2} + \frac{1}{2y^3}\right)^2}\, dy$$

$$= \int_1^2 \frac{y^3}{2} + \frac{1}{2} y^{-3}\, dy = \frac{y^4}{8} + \frac{1}{2} \cdot \frac{y^{-2}}{-2} \Big]_1^2 = \frac{y^4}{8} - \frac{1}{4} \cdot \frac{1}{y^2} \Big]_1^2$$

$$= \frac{16}{8} - \frac{1}{4} \cdot \frac{1}{4} - \left(\frac{1}{8} - \frac{1}{4}\right) = 2 - \frac{1}{16} + \frac{1}{8} = \boxed{\frac{33}{16}}$$

Example. Find the length of the arc of $y = 1 + 6x^{\frac{3}{2}}$ where $0 \le x \le 1$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + (9x^{1/2})^2}\, dx$$

$$\frac{dy}{dx} = 6 \cdot \frac{3}{2} x^{1/2} = 9x^{1/2}$$

$$= \int_0^1 \sqrt{1 + 81x}\, dx$$

$u = 1 + 81x$

$du = 81\, dx$

$\frac{1}{81} du = dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 1 + 81 = 82$

$$= \int_1^{82} \frac{1}{81}\sqrt{u}\, du = \frac{1}{81} \cdot \frac{2}{3} u^{3/2} \Big]_1^{82}$$

$$= \boxed{\frac{2}{243}\left[82^{3/2} - 1\right]}$$

Example. Find the length of the arc of $y = \ln(\sec x)$ where $0 \le x \le \frac{\pi}{4}$.

$$y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big]_0^{\pi/4} = \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0| = \boxed{\ln(\sqrt{2} + 1)}$$

Example. Find the length of the arc of $x = \frac{1}{3}\sqrt{y}(y-3)$ where $1 \le y \le 9$.

$x = \frac{1}{3}\sqrt{y}(y-3) = \frac{1}{3}y^{3/2} - y^{1/2} \Rightarrow \frac{dx}{dy} = \frac{1}{3}\cdot\frac{3}{2}y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$

$L = \int_1^9 \sqrt{1+\left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2}\,dy = \int_1^9 \sqrt{1+\frac{1}{4}\cdot y - 2\cdot\frac{1}{4} + \frac{1}{4}y^{-1}}\,dy$

$= \int_1^9 \sqrt{\frac{y}{4} + \frac{1}{2} + \frac{1}{4y}}\,dy = \int_1^9 \sqrt{\left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2}\,dy$

$= \int_1^9 \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\,dy = \frac{1}{2}\cdot\frac{2}{3}y^{3/2} + \frac{1}{2}\cdot 2y^{1/2}\Big]_1^9 = \frac{1}{3}y^{3/2} + y^{1/2}\Big]_1^9$

$= \frac{1}{3}\cdot 9^{3/2} + 9^{1/2} - \frac{1}{3} - 1 = \frac{1}{3}\cdot 27 + 3 - \frac{1}{3} - 1 = \boxed{\frac{32}{3}}$

The Arc Length Function

$s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\,dt$ is called the **arc length function** and it measures the arc length of a curve from a particular starting point to any other point on the curve.

Example. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking (1,1) as the starting point.

The ending point is variable call it x

$f(t) = t^2 - \frac{1}{8}\ln t \Rightarrow f'(t) = 2t - \frac{1}{8}\cdot\frac{1}{t}$

$s(x) = \int_1^x \sqrt{1+[f'(t)]^2}\,dt = \int_1^x \sqrt{1+\left(2t-\frac{1}{8t}\right)^2}\,dt$

$= \int_1^x \sqrt{1+4t^2-\frac{1}{2}+\frac{1}{64t^2}}\,dt = \int_1^x \sqrt{4t^2+1+\frac{1}{64t^2}}\,dt$

$= \int_1^x \sqrt{\left(2t+\frac{1}{8t}\right)^2}\,dt = \int_1^x 2t + \frac{1}{8}\cdot\frac{1}{t}\,dt = t^2 + \frac{1}{8}\ln|t|\Big]_1^x$

$= \boxed{x^2 + \frac{1}{8}\ln|x| - 1}$

Homework 11.1 Solutions: Arc Length

Prof. Leyda Almodóvar

Name: _____

1. Find the exact length of the curve.

   (a) $36y^2 = (x^2-4)^3$, $2 \le x \le 3$, $y \ge 0$

   $y^2 = \frac{(x^2-4)^3}{36}$, then $y = \frac{(x^2-4)^{\frac{3}{2}}}{6}$ and $\frac{dy}{dx} = \frac{3}{2}\frac{(x^2-4)^{\frac{1}{2}}\cdot 2x}{6} = \frac{x\sqrt{x^2-4}}{2}$

   $L = \int_2^3 \sqrt{1+\left(\frac{x\sqrt{x^2-4}}{2}\right)^2}\,dx = \int_2^3 \sqrt{1+\frac{x^2(x^2-4)}{4}}\,dx = \int_2^3 \sqrt{\frac{4+x^4-4x^2}{4}}\,dx$

   $= \frac{1}{2}\int_2^3 \sqrt{x^4-4x^2+4}\,dx = \frac{1}{2}\int_2^3 \sqrt{(x^2-2)^2}\,dx = \frac{1}{2}\int_2^3 (x^2-2)\,dx = \frac{1}{2}\left[\frac{x^3}{3}-2x\right]_2^3$

   $= \frac{27}{6} - 3 - \left(\frac{8}{6}-2\right) = \boxed{\frac{13}{6}}$

   (b) $x = \frac{y^4}{8} + \frac{1}{4y^2}$, $1 \le y \le 2$

   $\frac{dx}{dy} = \frac{4y^3}{8} - \frac{2}{4y^3} = \frac{y^3}{2} - \frac{1}{2y^3}$

   $L = \int_1^2 \sqrt{1+\left(\frac{y^3}{2}-\frac{1}{2y^3}\right)^2}\,dy = \int_1^2 \sqrt{1+\left(\frac{y^6}{4}-2\cdot\frac{1}{4}+\frac{1}{4y^6}\right)}\,dy = \int_1^2 \sqrt{\frac{y^6}{4}+\frac{1}{2}+\frac{1}{4y^6}}\,dy$

   $= \int_1^2 \sqrt{\left(\frac{y^3}{2}+\frac{1}{2y^3}\right)^2}\,dy = \int_1^2 \left(\frac{y^3}{2}+\frac{1}{2y^3}\right)dy = \frac{1}{2}\left[\frac{y^4}{4}-\frac{1}{4y^2}\right]_1^2 = \frac{1}{2}\left[\frac{2^4}{4}-\frac{1}{4\cdot 4}-\left(\frac{1}{4}-\frac{1}{4}\right)\right]$

   $= \frac{1}{2}\left(4-\frac{1}{16}\right) = \boxed{\frac{63}{32}}$

   (c) $y = \ln(\cos x)$, $0 \le x \le \frac{\pi}{3}$

   $\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

   $L = \int_0^{\frac{\pi}{3}} \sqrt{1+(-\tan x)^2}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\,dx$

$$= \int_0^{\frac{\pi}{3}} \sec x \, dx = \Big[\ln|\sec x + \tan x| \Big]_0^{\frac{\pi}{3}} = \ln|\sec \tfrac{\pi}{3} + \tan \tfrac{\pi}{3}| - \ln|\sec 0 + \tan 0|$$

$$= \ln|2 + \sqrt{3}| - \ln|1 + 0| = \boxed{\ln(2 + \sqrt{3})}$$

2. Find the length of the arc of the curve $x^2 = (y-4)^3$ from point $P = (1, 5)$ to point $Q = (8, 8)$.

$x = (y-4)^{\frac{3}{2}}$ and $\frac{dx}{dy} = \frac{3}{2}(y-4)^{\frac{1}{2}}$

$$L = \int_5^8 \sqrt{1 + \left[\tfrac{3}{2}(y-4)^{\frac{1}{2}}\right]^2} \, dy = \int_5^8 \sqrt{1 + \tfrac{9}{4}(y-4)} \, dy = \int_5^8 \sqrt{1 + \tfrac{9y}{4} - 9} \, dy = \int_5^8 \sqrt{-8 + \tfrac{9}{4}y} \, dy$$

Let $u = \frac{9}{4} - 8$, then $du = \frac{9}{4} dy$. If $y = 5$, $u = \frac{9}{4} \cdot 5 - 8 = \frac{13}{4}$ and if $y = 8$, then $u = \frac{9}{4} \cdot 8 - 8 = 10$. Hence,

$$\int_5^8 \sqrt{-8 + \tfrac{9}{4}y} \, dy = \tfrac{4}{9} \int_{\frac{13}{4}}^{10} \sqrt{u} \, du = \Big[\tfrac{4}{9} \cdot \tfrac{2}{3} u^{\frac{3}{2}} \Big]_{\frac{13}{4}}^{10} = \boxed{\frac{8}{27}\left[10\sqrt{10} - \frac{13\sqrt{13}}{8}\right]}$$

11.2 Area of a Surface of Revolution

A surface of revolution is formed when a curve is rotated about a line.

In the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the ~~x-axis as~~

$$\boxed{S = \int 2\pi y \, ds} \text{ for rotation about the } x\text{-axis}$$

$$\boxed{S = \int 2\pi x \, ds} \text{ for rotation about the } y\text{-axis}$$

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$ (either ds may be used).

Example. Determine the surface area of the solid obtained by rotating $y = \sqrt{9 - x^2}$, $-2 \le x \le 2$ about the x-axis.

Since we rotated about the x-axis $S = \int 2\pi y \, ds$

It's easier to find $\frac{dy}{dx}$ than $\frac{dx}{dy}$ so let $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$$\frac{dy}{dx} = \frac{1}{2}(9 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_{-2}^{2} y \sqrt{1 + \left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\,dx = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \cdot \sqrt{1 + \frac{x^2}{9 - x^2}}\,dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \cdot \sqrt{\frac{9 - x^2 + x^2}{9 - x^2}}\,dx = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \cdot \sqrt{\frac{9}{9 - x^2}}\,dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \cdot \frac{\sqrt{9}}{\sqrt{9 - x^2}}\,dx = 2\pi \int_{-2}^{2} 3\,dx = 2\pi [3x]_{-2}^{2} = 2\pi [6 - (-6)]$$

$$= \boxed{24\pi}$$

$x = y^3$

Example. Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the y-axis. Use both ds's to compute the surface area.

Solution #1: $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$, $y = x^{1/3} \Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-2/3}$

$$S = \int_a^b 2\pi x\,ds = 2\pi \int_1^8 x\sqrt{1 + \left(\frac{1}{3}x^{-2/3}\right)^2}\,dx = 2\pi \int_1^8 x\sqrt{1 + \frac{1}{9}x^{-4/3}}\,dx$$

$$= 2\pi \int_1^8 x\sqrt{\frac{9x^{4/3}+1}{9x^{4/3}}}\,dx = 2\pi \int_1^8 x\frac{\sqrt{9x^{4/3}+1}}{\sqrt{9x^{4/3}}}\,dx = 2\pi \int_1^8 x\frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}}\,dx$$

$$= \frac{2\pi}{3}\int_1^8 x^{1/3}\sqrt{9x^{4/3}+1}\,dx = \frac{2\pi}{3}\cdot\frac{1}{12}\int_{10}^{145}\sqrt{u}\,du$$

$$= \frac{\pi}{18}\cdot\frac{2}{3}u^{3/2}\Big]_{10}^{145}$$

$$= \boxed{\frac{\pi}{27}\left[145^{3/2} - 10^{3/2}\right]}$$

$u = 9x^{4/3} + 1$

$du = 9\cdot\frac{4}{3}x^{1/3}\,dx$

$\frac{1}{12}du = x^{1/3}\,dx$

$x = 1 \Rightarrow u = 9 + 1 = 10$

$x = 8 \Rightarrow u = 9\cdot 8^{4/3} + 1$
$= 9\cdot 16 + 1$
$= 145$

Solution #2: $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$, $y = x^{1/3} \Rightarrow x = y^3$, $\frac{dx}{dy} = 3y^2$

$$S = \int 2\pi x\,ds = 2\pi \int_1^2 x\sqrt{1 + (3y^2)^2}\,dy = 2\pi \int_1^2 y^3\sqrt{1 + 9y^4}\,dy$$

$$= \frac{2\pi}{36}\int_{10}^{145}\sqrt{u}\,du = \cdots = \boxed{\frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)}$$

↓ same integral as above

$u = 1 + 9y^4$

$du = 36y^3\,dy$

$\frac{1}{36}du = y^3\,dy$

$y = 1 \Rightarrow u = 1 + 9 = 10$

$y = 2 \Rightarrow u = 1 + 9\cdot 16 = 145$

Example. The arc of the parabola $y = x^2$ from (1,1) to (2,4) is rotated about the y-axis. Find the area of the resulting surface.

$$S = \int 2\pi x \, ds = 2\pi \int_1^2 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_1^2 x \sqrt{1 + (2x)^2} \, dx$$

$$= 2\pi \int_1^2 x \sqrt{1 + 4x^2} \, dx = \frac{2\pi}{8} \int_5^{17} \sqrt{u} \, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big]_5^{17}$$

$$= \boxed{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

$u = 1 + 4x^2$

$du = 8x \, dx$

$\frac{1}{8} du = x \, dx$

$x = 1 \Rightarrow u = 5$

$x = 2 \Rightarrow u = 17$

Example. Find the exact area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 2$ about the x-axis.

$$S = \int 2\pi y \, ds = 2\pi \int_0^2 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2} \, dx = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4} \, dx$$

$$= \frac{2\pi}{36} \int_1^{145} \sqrt{u} \, du$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big]_1^{145} = \boxed{\frac{\pi}{27}\left(145^{3/2} - 1\right)}$$

$u = 1 + 9x^4$

$du = 36x^3 \, dx$

$\frac{1}{36} du = x^3 \, dx$

$x = 0 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 1 + 9 \cdot 16 = 145$

Homework 11.2 Solutions: Area of a Surface of Revolution

Prof. Leyda Almodóvar

Name: _____

1. Find the exact area of the surface obtained by rotating the curve about the x-axis.

   (a) $y = \sqrt{5-x}$, $3 \le x \le 5$

   $S = \int 2\pi y \, ds$, $y = \sqrt{5-x}$, $\frac{dy}{dx} = \frac{1}{2}(5-x)^{-\frac{1}{2}} \cdot (-1) = -\frac{1}{2\sqrt{5-x}}$

   $S = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_3^5 2\pi\sqrt{5-x} \cdot \sqrt{1 + \left(-\frac{1}{2\sqrt{5-x}}\right)^2} \, dx = \int_3^5 2\pi\sqrt{5-x}\sqrt{1 + \frac{1}{4(5-x)}} \, dx$

   $= 2\pi \int_3^5 \sqrt{5-x} \cdot \sqrt{\frac{20-4x+1}{4(5-x)}} = 2\pi \int_3^5 \sqrt{5-x} \cdot \frac{\sqrt{20-4x+1}}{2\sqrt{5-x}} \, dx = \pi \int_3^5 \sqrt{21-4x} \, dx$

   Let $u = 21 - 4x$, then $du = -4\,dx$. If $x = 3$, $u = 21 - 4(3) = 12$, and if $x = 5$, then $u = 21 - 4(5) = 1$. Hence,

   $\pi \int_3^5 \sqrt{21-4x} \, dx = -\frac{\pi}{4} \int_9^1 \sqrt{u} \, du = \frac{\pi}{4} \int_1^9 u^{\frac{1}{2}} \, du = \left[\frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}}\right]_1^9 = \frac{\pi}{6}\left(9^{\frac{3}{2}} - 1\right)$

   $= \frac{\pi}{6}(9 \cdot 3 - 1) = \boxed{\frac{13\pi}{3}}$

   (b) $y = \sqrt{1+e^x}$, $0 \le x \le 1$

   $\frac{dy}{dx} = \frac{1}{2}(1+e^x)^{-\frac{1}{2}} \cdot e^x = \frac{e^x}{2\sqrt{1+e^x}}$

   $S = \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 2\pi\sqrt{1+e^x} \cdot \sqrt{1 + \frac{e^{2x}}{4(1+e^x)}} \, dx$

   $= \int_0^1 2\pi\sqrt{1+e^x} \cdot \sqrt{\frac{4+4e^x+e^{2x}}{4(1+e^x)}} \, dx = \int_0^1 2\pi\sqrt{1+e^x} \cdot \frac{\sqrt{4+4e^x+e^{2x}}}{\sqrt{4(1+e^x)}} \, dx$

   $= \int_0^1 \pi\sqrt{e^{2x}+4e^x+4} \, dx = \pi \int_0^1 \sqrt{(e^x+2)^2} \, dx = \pi \int_0^1 (e^x+2) \, dx$

   $= \pi\Big[(e^x+2x)\Big]_0^1 = \pi[(e+2) - e^0 - 2(0) \;= \boxed{\pi(e+1)}$

(c) $y = \frac{x^3}{6} + \frac{1}{2x}$, $\frac{1}{2} \le x \le 1$

$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2}$

$S = \int 2\pi y \, ds = \int_{\frac{1}{2}}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{\frac{1}{2}}^{1} 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \cdot \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} \, dx$

$= 2\pi \int_{\frac{1}{2}}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} \, dx = 2\pi \int_{\frac{1}{2}}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\left(\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}\right)} \, dx$

$= 2\pi \int_{\frac{1}{2}}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} \, dx = 2\pi \int_{\frac{1}{2}}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \cdot \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$

$= 2\pi \int_{\frac{1}{2}}^{1} \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right) dx = 2\pi \int_{\frac{1}{2}}^{1} \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$

$= 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_{\frac{1}{2}}^{1} = 2\pi\left(\frac{1}{72} + \frac{1}{6} - \frac{1}{8}\right) - 2\pi\left[\frac{(\frac{1}{2})^6}{72} + \frac{\frac{1}{4}}{6} - \frac{1}{8 \cdot \frac{1}{4}}\right] = \boxed{\frac{263\pi}{256}}$

(d) $x = 1 + 2y^2$, $1 \le y \le 2$

$\frac{dx}{dy} = 4y$

$S = \int 2\pi y \, ds = \int_1^2 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = 2\pi \int_1^2 y\sqrt{1 + (4y)^2} \, dy = 2\pi \int_1^2 y\sqrt{1 + 16y^2} \, dy$

Let $u = 16y^2 + 1$, then $du = 32y \, dy$. If $y = 1$, then $u = 17$ and if $y = 2$, then $u = 65$. Hence,

$2\pi \int_1^2 y\sqrt{1 + 16y^2} \, dy = \frac{\pi}{6} \int_{17}^{65} \sqrt{u} \, du = \frac{\pi}{16} \cdot \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{17}^{65} = \boxed{\frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)}$

2. The given curve is rotated about the y-axis. Find the area of the resulting surface.

(a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, $0 \le y \le 1$

$x^{\frac{2}{3}} = 1 - y^{\frac{2}{3}}$, then $x = (1 - y^{\frac{2}{3}})^{\frac{3}{2}}$, and $\frac{dx}{dy} = \frac{3}{2}(1 - y^{\frac{2}{3}})^{\frac{1}{2}} \cdot \left(-\frac{2}{3}y^{-\frac{1}{3}}\right) = \frac{-\sqrt{1-y^{\frac{2}{3}}}}{y^{\frac{1}{3}}}$

$S = \int 2\pi x \, ds = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_0^1 2\pi (1 - y^{\frac{2}{3}})^{\frac{3}{2}} \sqrt{1 + \frac{1-y^{\frac{2}{3}}}{y^{\frac{2}{3}}}} \, dy$

$= 2\pi \int_0^1 (1 - y^{\frac{2}{3}})^{\frac{3}{2}} \sqrt{\frac{1}{y^{\frac{2}{3}}}} \, dy = 2\pi \int_0^1 (1 - y^{\frac{2}{3}})^{\frac{3}{2}} \cdot \frac{1}{y^{\frac{1}{3}}} \, dy$

Let $u = 1 - y^{\frac{2}{3}}$, then $du = -\frac{2}{3}y^{-\frac{1}{3}} \, dy$, and $-\frac{3}{2} \, du = \frac{1}{y^{\frac{1}{3}}} \, dy$. If $y = 0$, $u = 1$, and if $y = 1$, $u = 0$. Hence,

$2\pi \int_0^1 (1-y^{\frac{2}{3}})^{\frac{3}{2}} \cdot \frac{1}{y^{\frac{1}{3}}}\,\mathrm{d}y = 2\pi\left(-\frac{3}{2}\right)\int_1^0 u^{\frac{3}{2}}\,\mathrm{d}u = 3\pi\int_0^1 u^{\frac{3}{2}}\,\mathrm{d}u = 3\pi\cdot\frac{2}{5}\left[u^{\frac{5}{2}}\right]_0^1 = \boxed{\frac{6\pi}{5}}$

(b) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x,\ 1 \le x \le 2$

$\frac{dy}{dx} = \frac{2}{4}\cdot x - \frac{1}{2}\cdot\frac{1}{x} = \frac{x}{2} - \frac{1}{2x}$

$S = \int 2\pi x\,\mathrm{d}s = \int_1^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\mathrm{d}x = \int_1^2 2\pi x\sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\,\mathrm{d}x$

$= 2\pi\int_1^2 x\sqrt{1+\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}}\,\mathrm{d}x = 2\pi\int_1^2 x\sqrt{\frac{x^2}{4}+\frac{1}{4x^2}+\frac{1}{2}}$

$= 2\pi\int_1^2 x\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,\mathrm{d}x = 2\pi\int_1^2 x\left(\frac{x}{2}+\frac{1}{2x}\right)\mathrm{d}x = \pi\int_1^2 x^2+1\,\mathrm{d}x = \pi\left[\frac{x^3}{3}+x\right]_1^2$

$= \pi\left(\frac{8}{3}+2-\frac{1}{3}-1\right) = \boxed{\frac{10\pi}{3}}$

Midterm 1 MTH-126

Prof. Leyda Almodóvar

Name: _____

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, you may use an additional blank sheet. You must show your work on every problem. Answers with no work or justification shown will get no credit.

Question	Points	Score
1	10	
2	11	
3	11	
4	11	
5	11	
6	12	
7	12	
8	11	
9	11	
Total:	100	

1. (10 points) Sketch the region enclosed by the curves $y = 12 - x^2$, $y = x^2 - 6$. Then, find the area of the region.

2. (11 points) Find the volume of the solid obtained by rotating the region bounded by the curves $y^2 = x$, $x = 2y$ about the y-axis. Sketch the **region**, the **solid**, and a **typical disk or washer**.

3. (11 points) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = x^3$, $y = 0$, $x = 1$, $x = 2$ about the y-axis.

4. (11 points) Evaluate the integral $\int_1^2 x^2 \ln x \, \mathrm{d}x$.

5. (11 points) Evaluate the integral $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, \mathrm{d}x$.

6. (12 points) Evaluate the integral $\int \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x$.

7. (12 points) Evaluate the integral $\int \frac{x^2-x+6}{x^3+3x}\,\mathrm{d}x$

8. (11 points) Evaluate the integral $\int_1^\infty \frac{\ln x}{x}\,\mathrm{d}x$.

9. (11 points) Evaluate the integral $\int_0^1 \frac{1}{\sqrt{1-x}}\,\mathrm{d}x$.

Review Exam 1

7.6

29) $\int_2^\infty \frac{1}{v^2+2v-3}\,dv = \lim_{t\to\infty}\int_2^t \frac{1}{v^2+2v-3}\,dv$

$\left[\frac{1}{v^2+2v-3} = \frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}\right] \cdot (v+3)(v-1)$

$1 = A(v-1) + B(v+3)$

$v=1$: $1 = A\cdot 0 + B\cdot 4 \Rightarrow B = 1/4$

$v=-3$: $1 = A(-4) + B\cdot 0 \Rightarrow A = -1/4$

$= \lim_{t\to\infty}\int_2^t \frac{-1/4}{v+3} + \frac{1/4}{v-1}\,dv$

$= \lim_{t\to\infty}\left[\frac{-1}{4}\int_2^t \frac{1}{v+3}\,dv + \frac{1}{4}\int_2^t \frac{1}{v-1}\,dv\right]$

$= \lim_{t\to\infty}\left[-\frac{1}{4}\cdot\ln|v+3| + \frac{1}{4}\cdot\ln|v-1|\right]_2^t$

$= \lim_{t\to\infty}\left[\underbrace{-\frac{1}{4}\cdot\ln|t+3|}_{-\infty} + \underbrace{\frac{1}{4}\cdot\ln|t-1|}_{\infty} + \frac{1}{4}\cdot\ln 5 - \frac{1}{4}\cdot\ln 1\right]$

$$= \lim_{t\to\infty}\left[\frac{1}{4}\cdot\ln\left|\frac{t-1}{t+3}\right| + \frac{1}{4}\cdot\ln 5 - 0\right]$$

$\frac{t-1}{t+3}$: $\frac{\infty}{\infty}$

$$\overset{\text{H}}{=} \lim_{t\to\infty}\left[\frac{1}{4}\cdot\ln\left|\frac{1-0}{1+0}\right| + \frac{1}{4}\cdot\ln 5\right]$$

$$= 0 + \frac{1}{4}\cdot\ln 5 = \boxed{\frac{1}{4}\cdot\ln 5}$$

$$\int_0^\infty \sin\theta\, e^{\cos\theta}\, d\theta = \lim_{t\to\infty}\int_0^t \sin\theta\, e^{\cos\theta}\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= \lim_{t\to\infty}\left[-\int_1^{\cos t} e^u\, du\right]$$

$$= \lim_{t\to\infty}\left[e^u\right]_1^{\cos t} = \lim_{t\to\infty}\left[-e^{\cos t} + e^1\right] = \text{D.N.E.}$$

Divergent

2e) $\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{-1/x}}{x^2}\,dx$

$u = -\frac{1}{x}$

$du = \frac{1}{x^2}dx$

$= \lim_{t\to\infty}\int_{-1}^{-\frac{1}{t}} e^u\,du$

$= \lim_{t\to\infty}\left[e^u\right]_{-1}^{-1/t} = \lim_{t\to\infty}\left[e^{-1/t} - e^{-1}\right]$

$= e^0 - e^{-1} = \boxed{1-e^{-1}}$ convergent

2d) $\int_{-\infty}^{\infty} y^3 - 3y^2\,dy$

$\int_{-\infty}^{\infty} y^3-3y^2\,dy = \int_{-\infty}^{0} y^3-3y^2\,dy + \int_0^\infty y^3-3y^2\,dy$

$\int_{-\infty}^{0} y^3-3y^2\,dy = \lim_{t\to-\infty}\int_t^0 y^3-3y^2\,dy$

$= \lim_{t\to-\infty}\left[\frac{y^4}{4} - y^3\right]_t^0 = \lim_{t\to-\infty}\left[0-0-\underbrace{\frac{t^4}{4}}_{-\infty} + \underbrace{t^3}_{-\infty}\right]$

$= -\infty$ Divergent

7.5 1h) $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$u = 1+(\ln x)^2$

$du = 2\cdot\ln x\cdot\frac{1}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\int \frac{u}{\sqrt{1+u^2}}\,du$

$U = 1+u^2$

$dU = 2u\,du$

1b) $\int t\sin t\cos t\,dt = \frac{1}{2}\int t\sin 2t\,dt$

$\sin t\cos t = \frac{1}{2}\sin 2t$

$= \frac{1}{2}\left[-\frac{t\cos 2t}{2} + \int \frac{\cos 2t}{2}\,dt\right]$

$u = t \qquad dv = \sin 2t\,dt$

$du = dt \qquad v = -\frac{\cos 2t}{2}$

$= -\frac{1}{4}\cdot t\cos 2t + \frac{1}{4}\cdot\frac{\sin(2t)}{2} + C$

$= \boxed{-\frac{1}{4}\cdot t\cos 2t + \frac{1}{8}\cdot\sin 2t + C}$

12) $\int \frac{(x-1)e^x}{x^2}\,dx = -\frac{(x-1)e^x}{x} + \int \frac{1}{x}\left[e^x + (x-1)e^x\right]dx$

$u = (x-1)e^x \qquad dv = \frac{1}{x^2}\,dx$

$du = \left[e^x + (x-1)e^x\right]dx \qquad v = -\frac{1}{x}$

$= -\frac{(x-1)e^x}{x} + \int \frac{e^x}{x} + \frac{(x-1)e^x}{x}\,dx$

$= -\frac{(x-1)e^x}{x} + \int \frac{e^x}{x} + e^x - \frac{e^x}{x}\,dx$

$= \boxed{-\frac{(x-1)e^x}{x} + e^x + C}$

or $\int \frac{(x-1)e^x}{x^2}\,dx = \int \frac{1}{x}\cdot e^x\,dx - \int \frac{e^x}{x^2}\,dx$

7.5 i) $\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx$

$x^3+x^2+x+1 = x^2(x+1)+x+1 = (x+1)(x^2+1)$

$$\left[\frac{3x^2+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right](x+1)(x^2+1)$$

$3x^2+1 = A(x^2+1)+(Bx+C)(x+1)$

$= Ax^2+A+Bx^2+Bx+Cx+C$

$= (A+B)x^2+(B+C)x+A+C$

$A+B=3$

$A+C=1$

$B+C=0 \Rightarrow C=-B$

$$\begin{array}{r} A+B=3 \\ +\ A-B=1 \\ \hline 2A=4 \end{array} \Rightarrow A=2$$

$\Rightarrow B=1 \Rightarrow C=-1$

$$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx = \int_0^1 \frac{2}{x+1} + \frac{x-1}{x^2+1}\,dx$$

split!

$$= 2\int_0^1 \frac{1}{x+1}\,dx + \int_0^1 \frac{x}{x^2+1}\,dx - \int_0^1 \frac{1}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= 2\ln|x+1|\Big]_0^1 + \frac{1}{2}\int_1^2 \frac{1}{u}\,du - \left[\tan^{-1}x\right]_0^1$$

$$= 2\ln 2 - 2\ln 1 + \frac{1}{2}\ln|u|\Big]_1^2 - \tan^{-1}(1) + \tan^{-1}(0)$$

$$= 2\ln 2 - 0 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 - \pi/4 + 0$$

$$= \boxed{\frac{5}{2}\ln 2 - \pi/4}$$

(c) $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \quad \Big](x-3)(x-2)$$

$$x-4 = A(x-2) + B(x-3)$$

$x=2$: $\quad -2 = B(-1) \Rightarrow B = 2$

$x=3$: $\quad -1 = A$

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \frac{-1}{x-3} + \frac{2}{x-2}\,dx$$

$$= -\ln|x-3| + 2\ln|x-2| \Big]_0^1$$

$$= -\ln|1-3| + 2\ln|1-2| + \ln|-3| - 2\ln|-2|$$

$$= -\ln 2 + 2\cdot\ln 1 + \ln 3 - 2\ln 2$$

$$= \boxed{-3\ln 2 + \ln 3}$$

$$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{x^2-2x+1}$$

$$\begin{array}{r} x^2 \\ x^2-2x+1 \;\overline{)\; x^4-2x^3+x^2+2x-1} \\ -(x^4-2x^3+x^2) \\ \hline 2x-1 \end{array}$$

$$\frac{2x-1}{x^2-2x+1} = \frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$\frac{\quad}{(x+3)^4} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4}$$

$x^2 = x \cdot x$

↓ not an irred. quad.

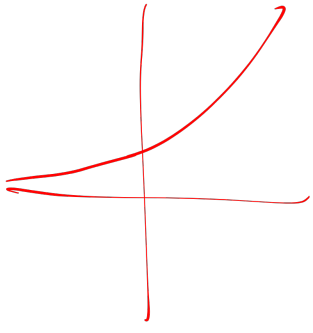
$$\frac{\quad}{x^2} = \frac{A}{x} + \frac{B}{x^2}$$

7.6 24) $\int_{-\infty}^{0} 2^r dr = \lim_{t\to-\infty} \int_t^0 2^r dr$

$= \lim_{t\to-\infty} \left[\frac{2^r}{\ln 2}\right]_t^0$

$= \lim_{t\to-\infty} \left[\frac{2^0}{\ln 2} - \frac{2^t}{\ln 2}\right]$

$= \frac{1}{\ln 2}$ convergent



$\int_{\pi/6}^{\pi/3} \frac{\sin\theta\cot\theta}{\sec\theta} d\theta = \int_{\pi/6}^{\pi/3} \frac{\sin\theta \cdot \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}} d\theta$

$= \int_{\pi/6}^{\pi/3} \cos^2\theta \, d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} 1+\cos 2\theta \, d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/3} = \frac{1}{2}\left[\pi/3 + \frac{\sin 2\pi/3}{2} - \pi/6 - \frac{\sin\pi/3}{2}\right]$

$= \frac{1}{2}\left[\pi/6 + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right] = \boxed{\frac{\pi}{12}}$

7.4(b)

$$\frac{x^2}{x^2+x+6} = 1 - \frac{(x+6)}{x^2+x+6} = 1 - \frac{Ax+B}{x^2+x+6}$$

$$\begin{array}{r} 1 \\ x^2+x+6 \overline{)\, x^2} \\ \underline{-(x^2+x+6)} \\ -x-6 \end{array}$$

Sample Exam 2 - MTH 126

Prof. Leyda Almodóvar

Name: _____

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, you may use an additional blank sheet. You must show your work on every problem. Answers with no work or justification shown will get no credit.

Question	Points	Score
1	6	
2	16	
3	42	
4	12	
5	12	
6	12	
Total:	100	

1. (6 points) Determine whether the sequence $a_n = \frac{5n^2+1}{n^2+3n+5}$ converges or diverges. If it converges, find the limit.

2. (16 points) Determine whether the series is convergent or divergent. If it is convergent, find its sum. Clearly state the test or criteria being used to test the series.

   (a) $\sum_{n=1}^{\infty} \frac{1}{\pi^n}$

   (b) $\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$

3. (42 points) Determine whether the series is convergent or divergent. Clearly state the test or criteria being used to test the series.

(a) $\sum_{n=1}^{\infty} 5n^{-\frac{3}{2}}$

(b) $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

(d) $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

(e) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

(f) $\sum_{k=1}^{\infty} ke^{-k^2}$

4. (12 points) Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$.

5. (12 points) Find a power series representation for $f(x) = \frac{5}{1-4x^2}$ and determine the interval of convergence.

6. (12 points) Find the MacLaurin series for $f(x) = \ln(1+x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

Review Exam 2

8.3 2c) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

- $f(x) = \frac{\ln x}{x^2} > 0$ when $\ln x > 0 \Rightarrow x > 1$
- f is continuous since $x > 2$ (cannot divide by 0)
- $f'(x) = \frac{(\ln x)'(x^2) - (x^2)'(\ln x)}{(x^2)^2} = \frac{\frac{1}{x} \cdot x^2 - 2x \cdot \ln x}{x^4}$

$= \frac{x - 2x\ln x}{x^4} < 0$ when $x - 2x\ln x < 0$

$x(1 - 2\ln x) < 0$

$\Rightarrow x > 0$ and $1 - 2\ln x < 0$

$1 < 2\ln x$

$\frac{1}{2} < \ln x$

$e^{1/2} < x$

$\Rightarrow f$ is ultimately decreasing

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[\left[\frac{-1}{x} \ln x \right]_2^t + \int_2^t \frac{1}{x^2}\,dx \right]$$

$u = \ln x \quad dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx \quad v = -\frac{1}{x}$

$$= \lim_{t\to\infty} \left[-\frac{1}{x}\ln x - \frac{1}{x} \right]_2^t = \lim_{t\to\infty} \left[\frac{-\ln t}{t} - \frac{1}{t} + \frac{\ln 2}{2} + \frac{1}{2} \right]$$

($\frac{-\ln t}{t} \to \frac{\infty}{\infty}$; $-\frac{1}{t} \to 0$)

$$\overset{\text{H}}{=} \lim_{t\to\infty} \left[\frac{-1/t}{1} + \frac{\ln 2}{2} + \frac{1}{2} \right] = \frac{\ln 2}{2} + \frac{1}{2} \Rightarrow \text{integral converges}$$

($-1/t \to 0$)

$\Rightarrow \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test

§9 3a) Find a power series rep. for $x^2\tan^{-1}(x^3)$

We found $\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $R=1$

$$\tan^{-1}(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

$$x^2\cdot\tan^{-1}(x^3) = x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+5}}{2n+1}}$$

$\boxed{R=1}$

3b) $f(x) = \left(\frac{x}{2-x}\right)^3 = \frac{x^3}{(2-x)^3} = x^3 \cdot \frac{1}{(2-x)^3}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

$(2-x)^{-1}$

$$-(2-x)^{-2}\cdot(-1) = \frac{1}{(2-x)^2} = \frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{n x^{n-1}}{2^n}$$

$(2-x)^{-2}$

$\left|\frac{x}{2}\right| < 1$

$|x| < 2$

$\boxed{I=(-2,2)}$ $\boxed{R=2}$

$$-2(2-x)^{-3}\cdot(-1) = \frac{2}{(2-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^n}$$

$$\frac{1}{(2-x)^3} = \frac{1}{4}\cdot\sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^n}$$

$$f(x) = x^3\cdot\frac{1}{(2-x)^3} = \frac{x^3}{4}\cdot\sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2^n}$$

$$= \boxed{\sum_{n=2}^{\infty}\frac{n(n-1)x^{n+1}}{2^{n+2}}} = \sum_{n=0}^{\infty}\frac{(n+2)(n+1)x^{n+3}}{2^{n+4}}$$

replace n with n+2

8.8 2e) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n\,5^n}$ Find I and R.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)\cdot 5^{n+1}}\cdot\frac{n\cdot 5^n}{(-1)^{n-1}x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-x\cdot n}{(n+1)\cdot 5}\right| = \left|\frac{-x}{5}\right|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$\overset{\#}{=} \left|\frac{-x}{5}\right|\cdot\lim_{n\to\infty}\frac{1}{1+0} = \left|\frac{-x}{5}\right| < 1 \Rightarrow |x| < 5$$

$R = 5$

$-5 < x < 5$ Test endpoints

$x=-5$: $\sum_{n=0}^{\infty} \frac{(-1)^n(-5)^n}{n\cdot 5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(-1)^n\cdot 5^n}{n\cdot 5^n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$ p-series $p=1$ divergent

$x=5$: $\sum_{n=0}^{\infty} \frac{(-1)^n\cdot 5^n}{n\cdot 5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$

$b_n = \frac{1}{n} \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow$ decreasing

$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ convergent by AST

$\boxed{I = (-5, 5]}$

8.7 1d) $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

Root Test

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^{2n}}{(1+n)^{3n}}\right|} = \lim_{n\to\infty} \left|\frac{n^{2n}}{(1+n)^{3n}}\right|^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left|\frac{n^2}{(1+n)^3}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{(\frac{1}{n}+1)^3}\right| = \frac{0}{1} = 0 < 1$ → criteria

$\Rightarrow$ Converges by the Root Test

conclusion — test

8.6 1d) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-2)^n}\right| = \lim_{n\to\infty}\left|-2\cdot\left(\frac{n}{n+1}\right)^2\right|$$

$$= \lim_{n\to\infty}\left|-2\cdot\left(\frac{1}{1+1/n}\right)^2\right| = 2 > 1$$

$\Rightarrow$ diverges by the Ratio Test

8.8 2f) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$ Find R and I.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}\cdot n!}{(n+1)\cdot n!\, x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0 < 1$$

$R = \infty$

$I = (-\infty, \infty)$

8.9 4b) $\int \frac{\tan^{-1}x}{x}\,dx$

$$\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}, \quad R=1$$

$$\frac{\tan^{-1}x}{x} = \frac{1}{x}\cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)}, \quad R=1$$

$$\int \frac{\tan^{-1}x}{x}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)}$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}} \quad \boxed{R=1}$$

4(a) $\int \frac{t}{1+t^3}\,dt$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{t}{1+t^3} = t\cdot\frac{1}{1-(-t^3)} = t\cdot\sum_{n=0}^{\infty}(-t^3)^n = t\cdot\sum_{n=0}^{\infty}(-1)^n t^{3n}$$

$$= \sum_{n=0}^{\infty}(-1)^n t^{3n+1}$$

$$\int \frac{t}{1+t^3}dt = \int \sum_{n=0}^{\infty}(-1)^n t^{3n+1}\,dt = \boxed{C + \sum_{n=0}^{\infty}\frac{(-1)^n t^{3n+2}}{3n+2}}$$

Need $|-t^3|<1$

$|t|<1$

$\boxed{R=1}$

8.8 2g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$ Find R and I.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(2n+1)2^{n+1}}\cdot\frac{(2n-1)2^n}{(-1)^n(x-1)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{-(x-1)(2n-1)}{(2n+1)\cdot 2}\right| = \left|-\frac{(x-1)}{2}\right|\cdot\lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\right|$$

$$\stackrel{\text{H}}{=}\left|-\frac{(x-1)}{2}\right|\cdot\lim_{n\to\infty}\frac{2+0}{2+0} = \left|\frac{-(x-1)}{2}\right| < 1$$

$\Rightarrow |x-1| < 2$ $\quad R=2$

$\Rightarrow -2 < x-1 < 2$

$-1 < x < 3$

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{(2n-1)\cdot 2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot(-1)^n\cdot 2^n}{(2n-1)\cdot 2^n} = \sum_{n=0}^{\infty}\frac{1}{2n-1}$

Compare to $\sum_{n=0}^{\infty}\frac{1}{2n} = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{1}{n}$ p-series $p=1$ divergent

$\frac{1}{2n-1} > \frac{1}{2n} \Rightarrow \sum\frac{1}{2n-1}$ diverges by Comparison Test

$x=3$: $\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n}{(2n-1)\cdot 2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n-1}$ $\quad b_n = \frac{1}{2n-1}$

$\frac{1}{2n+1} < \frac{1}{2n-1} \Rightarrow$ dec.

$\lim_{n\to\infty}\frac{1}{2n-1} = 0 \Rightarrow$ convergent by AST

$I = (-1, 3]$

8.10 3c) $f(x)=\cos x,\ a=\pi/2$

$$\cos x=\sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n$$

$f^{(0)}(x)=f(x)=\cos x$	$f(\pi/2)=\cos\pi/2=0$
$f'(x)=-\sin x$	$f'(\pi/2)=-\sin\pi/2=-1$
$f''(x)=-\cos x$	$f''(\pi/2)=0$
$f'''(x)=\sin x$	$f'''(\pi/2)=1$
$f^{(4)}(x)=\cos x$	

$$\cos x=\sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n=\frac{-1}{1!}(x-\pi/2)+\frac{1}{3!}(x-\pi/2)^3$$

$$-\frac{1}{5!}(x-\pi/2)^5+\frac{1}{7!}(x-\pi/2)^7+\cdots$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-\pi/2)^{2n+1}}{(2n+1)!}}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-\pi/2)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^{n+1}(x-\pi/2)^{2n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{-(x-\pi/2)^2\cdot(2n+1)!}{(2n+3)(2n+2)\cdot(2n+1)!}\right|=\lim_{n\to\infty}\left|\frac{-(x-\pi/2)^2}{(2n+3)(2n+2)}\right|$$

$=0<1$ $\boxed{R=\infty}$

8.7(h) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$

Compare to $\sum_{n=1}^{\infty} \frac{\sqrt{n^4}}{n^3} = \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$

p-series $p=1$ $\Rightarrow$ divergent

LCT:

$$\lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^3+n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n\cdot\frac{\sqrt{n^4+1}}{n^3}}{\frac{n^3}{n^3}+\frac{n}{n^3}}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{n^4+1}}{n^2}}{1+1/n^2} \quad \left(\frac{1}{n^2} = \sqrt{\frac{1}{n^4}}\right) = \lim_{n\to\infty} \frac{\sqrt{\frac{n^4+1}{n^4}}}{1+1/n^2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n^4}}{1+1/n^2} = 1 > 0$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ diverges by LCT

Sample Exam 3 - MTH 126

Prof. Leyda Almodóvar

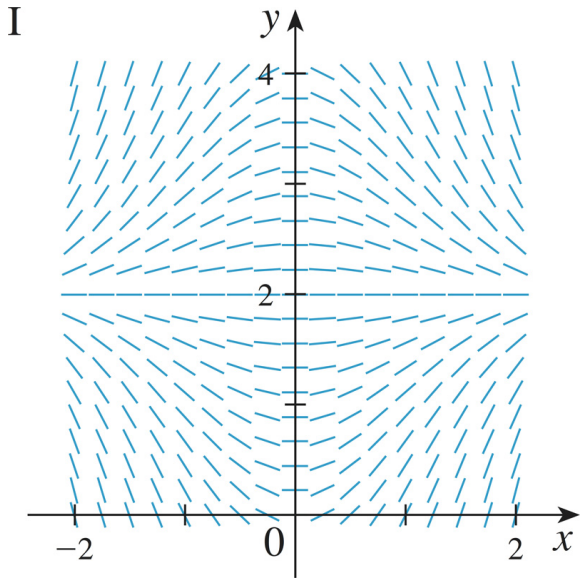
Name: _____

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, you may use an additional blank sheet. You must show your work on every problem. Answers with no work or justification shown will get no credit.

Question	Points	Score
1	5	
2	5	
3	10	
4	10	
5	10	
6	14	
7	10	
8	12	
9	12	
10	12	
Total:	100	

1. (5 points) Verify that all members of the family of functions $y = \frac{\ln x + C}{x}$ are solutions of the differential equation $x^2y' + xy = 1$. Note that C is constant.

2. (5 points) A direction field for the differential equation $y' = x(2 - y)$ is shown.



(a) Sketch the graphs of the solutions that satisfy the given initial conditions $y(0) = 3$ **and** $y(1) = 1$.

(b) Find all the equilibrium solutions.

3. (10 points) Solve the initial-value problem $\frac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$.

4. (10 points) Solve the initial-value problem $x^2y' + 2xy = \ln x$, $y(1) = 2$.

5. (10 points) A curve is described by parametric equations $x = 3\cos t + 1$ and $y = 3\sin t$.
   (a) Sketch the parametric curve. Clearly indicate the direction of motion.
   (b) Eliminate the parameter to find an algebraic equation.

6. (14 points) Given the parametric equations $x = t^2 + 1$, $y = e^t - 1$,

(a) Find dy/dx and d^2y/dx^2.

(b) Find the points on the curve where the tangent is horizontal or vertical.

(c) For which values of t is the curve concave upward?

7. (10 points) Find the points (r, θ) on the polar curve $r = 3\cos\theta$ where the tangent line is horizontal.

8. (12 points) (a) Sketch the curves $r = 4\sin\theta$ and $r = 2$.

(b) Find the area of the region that lies inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$. Show this region on your graph.

9. (12 points) Find the exact length of the curve $x = \frac{y^4}{8} + \frac{1}{4y^2}$, $1 \leq y \leq 2$.

10. (12 points) Find the exact area of the surface obtained by rotating the curve $y = \sqrt{5 - x}$, $3 \leq x \leq 5$ about the x-axis.

Review Exam 3

MTH-126 A

10.1 #2 $x = \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$

(b) $y - 1 = -\cos t$

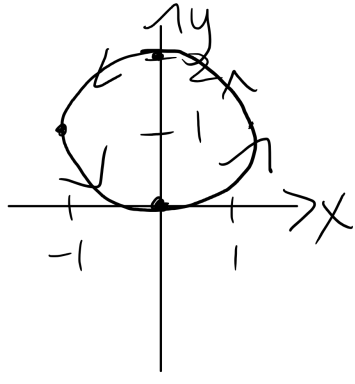
$(y-1)^2 = \cos^2 t$

$\cos^2 t + \sin^2 t = 1$

$(y-1)^2 + x^2 = 1$ ← circle centered at $(0,1)$ radius 1

(a)

t	$x = \sin t$	$y = 1 - \cos t$
0	0	0
$\pi/2$	1	1
π	0	2
$3\pi/2$	-1	1



3 $x=\sin\frac{1}{2}\theta$, $y=\cos\frac{1}{2}\theta$, $-\pi\le\theta\le\pi$

(a) $\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta=1$

$x^2+y^2=1$ ← circle centered at (0,0) radius 1

(b)

Q: Are we getting the whole circle?

If $-\pi\le\theta\le 0$

$\theta=-\pi$ $x=\sin\frac{1}{2}(-\pi)=\sin(-\pi/2)=-1 \Rightarrow -1\le x\le 0$

$\theta=0$ $x=\sin 0=0$ ←

$\theta=-\pi$, $y=\cos\frac{1}{2}(-\pi)=0 \Rightarrow 0\le y\le 1$

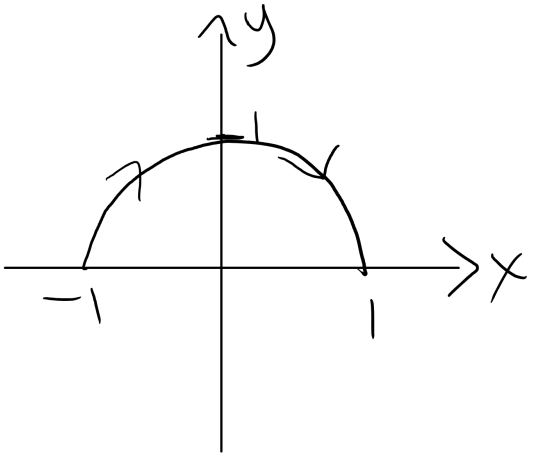
$\theta=0$, $y=\cos 0=1$

If $0\le\theta\le\pi$

$\theta=\pi$ $x=\sin\frac{1}{2}\pi=1 \Rightarrow 0\le x\le 1$

$y=\cos\frac{1}{2}\pi=0 \Rightarrow 0\le y\le 1$

⇒ A: No, we get the upper half



$\frac{dx}{d\theta} = \frac{1}{2}\cos\frac{1}{2}\theta$

$-\pi \le \theta \le 0, \quad \frac{dx}{d\theta} > 0$

$0 \le \theta \le \pi, \quad \frac{dx}{d\theta} > 0$

As θ increases $\Rightarrow$ Left to right

x increases

10.2 #5 Area of the surface obtained by rotating $x = t^3, y = t^2, 0 \le t \le 1$ about the x-axis.

$$S = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^1 t^2 \cdot \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

$$= 2\pi \int_0^1 t^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$= 2\pi \int_0^1 t^2 \sqrt{t^2(9t^2+4)}\, dt$$

$$= 2\pi \int_0^1 t^2 \cdot \sqrt{t^2} \cdot \sqrt{9t^2+4}\, dt$$

$$= 2\pi \int_0^1 t^3 \sqrt{9t^2+4}\, dt$$

$u = 9t^2 + 4 \Rightarrow \frac{u-4}{9} = t^2$

$du = 18t\,dt \Rightarrow \frac{1}{18}du = t\,dt$

$$= \frac{2\pi}{18}\int_4^{13} \left(\frac{u-4}{9}\right)\sqrt{u}\,du$$

$$= \frac{\pi}{9}\int_4^{13} \frac{u^{3/2}}{9} - \frac{4}{9}u^{1/2}\,du$$

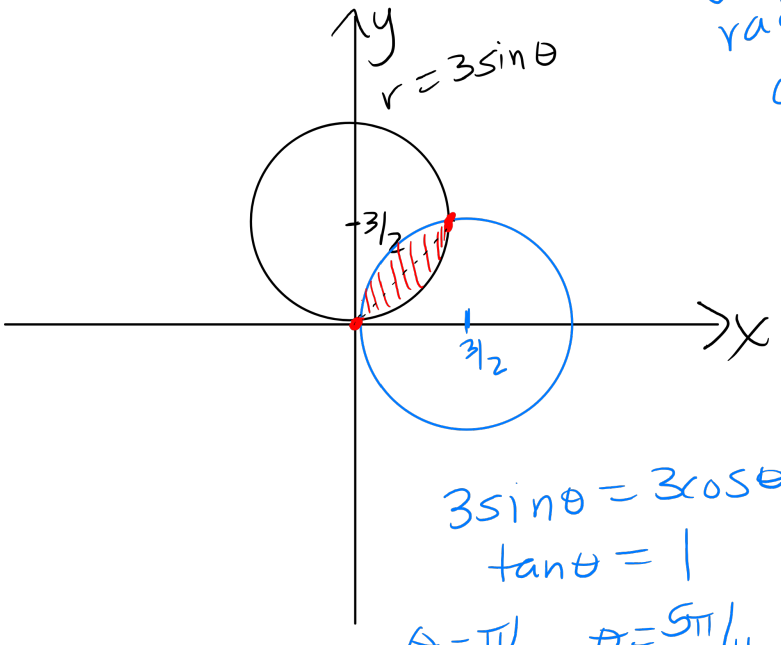
$$= \pi/81\left[\frac{2}{5}u^{5/2} - 4\cdot\frac{2}{3}u^{3/2}\right]_4^{13}$$

$$= \pi/81\left[\frac{2}{5}\cdot 13^{5/2} - \frac{8}{3}13^{3/2} - \frac{2}{5}\underbrace{4^{5/2}}_{32} + \frac{8}{3}\underbrace{4^{3/2}}_{8}\right]$$

10.4 #4 Find the area of the region that lies inside both curves $r = 3\sin\theta$, $r = 3\cos\theta$

$r = 2a\cos\theta$ or $2a\sin\theta$

radius $3/2$
center $(3/2, 0)$



$3\sin\theta = 3\cos\theta$

$\tan\theta = 1$

$\theta = \pi/4, \quad \theta = 5\pi/4$

$$A = 2\cdot\int_0^{\pi/4} \frac{1}{2}(3\sin\theta)^2\,d\theta = \int_0^{\pi/4} 9\sin^2\theta\,d\theta$$

$$= 9\int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/4}$$

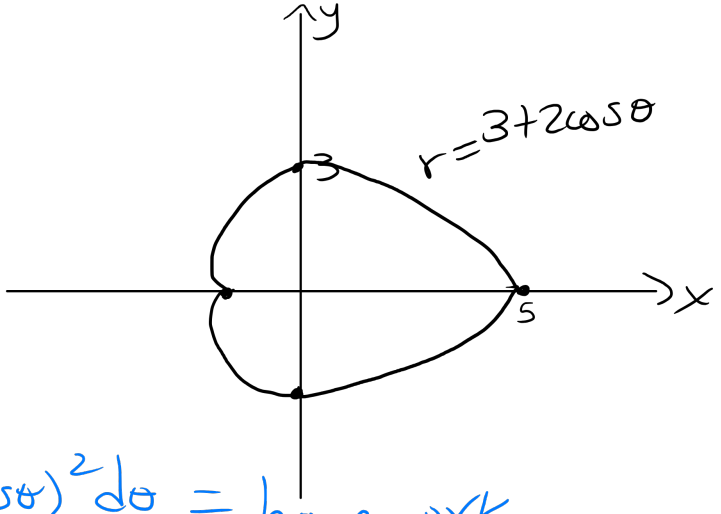
$$= \frac{9}{2}\left[\pi/4 - \frac{\sin\pi/2}{2} - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{9}{2}\left[\pi/4 - \frac{1}{2}\right] = \boxed{\frac{9\pi}{8} - \frac{9}{4}}$$

1b) Sketch the curve $r = 3 + 2\cos\theta$ and find the area that it encloses

Limaçon without a loop

θ	$r = 3+2\cos\theta$
0	5
$\pi/2$	3
π	1
$3\pi/2$	3



$$A = \int_0^{2\pi} \frac{1}{2}(3+2\cos\theta)^2 d\theta = \text{homework}$$

11.2 1d) $x = 1 + 2y^2$, $1 \le y \le 2$, rotating about the x-axis

$$S = \int 2\pi y\, ds = 2\pi \int_1^2 y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_1^2 y\sqrt{1+(4y)^2}\, dy = 2\pi \int_1^2 y\sqrt{1+16y^2}\, dy$$

$u = 1 + 16y^2$

$du = 32y\, dy$

$$= \frac{2\pi}{32}\int_{17}^{65} \sqrt{u}\, du = \frac{\pi}{16}\cdot\frac{2}{3}u^{3/2}\Big]_{17}^{65}$$

$$= \frac{\pi}{24}\left[65^{3/2} - 17^{3/2}\right]$$

11.2 2b) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \le x \le 2$, rotating about y-axis

$$S = \int 2\pi x \, ds = 2\pi \int_1^2 x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{x}$

$$= 2\pi \int_1^2 x \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x \sqrt{1 + \frac{x^2}{4} - 2\cdot\frac{x}{4x} - \frac{1}{4x^2}}\, dx$$

$-1/2$

$$= 2\pi \int_1^2 x \sqrt{\frac{x^2}{4} + \frac{1}{2} - \frac{1}{4x^2}}\, dx$$

$$= 2\pi \int_1^2 x \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x\left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^2 \frac{x^2}{2} + \frac{1}{2}\, dx$$

$$= \pi\left[\frac{x^3}{3} + x\right]_1^2 = \pi\left[\frac{8}{3} + 2 - \frac{1}{3} - 1\right]$$

$$= \boxed{\frac{10\pi}{3}}$$

1c) $y = \frac{x^3}{6} + \frac{1}{2x}$, $\frac{1}{2} \le x \le 1$, rotate about x-axis

$$S = \int 2\pi y \, ds = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} \, dx$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} \, dx$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} \, dx$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{x^3}{6} + \frac{1}{2x}\right) \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} \, dx$$

= homework

9.4 4a) 400 fish ← $P(0)=400$

$M=10000$ $P(1)=1200$ ←

Want: An expression for $P(t)$

$$P(t)=\frac{M}{1\pm e^{-C}e^{-kt}}$$

$$\pm e^{-C}=\frac{M-P(0)}{P(0)}=\frac{10000-400}{400}$$

$$=\frac{9600}{400}=24$$

$$=\frac{10000}{1+24e^{-kt}}$$

$$1200=\frac{10000}{1+24e^{-k}} \Rightarrow 1+24e^{-k}=\frac{10000}{1200}=\frac{25}{3}$$

$$\Rightarrow 24e^{-k}=25/3-1 \Rightarrow 24e^{-k}=22/3$$

$$e^{-k}=\frac{22}{3}\cdot\frac{1}{24}=\frac{11}{36} \Rightarrow -k=\ln(11/36)$$

$$k=-\ln(11/36)$$

$$P(t)=\frac{10000}{1+24e^{\ln(11/36)t}}$$ fish

10.3 (6a) $r = 3\cos\theta$

Find points on curve where the tangent is horizontal or vertical. (r, θ)

Horizontal tangent when $\frac{dy}{d\theta} = 0$

$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$= (-3\sin\theta)\sin\theta + (3\cos\theta)\cos\theta$

$= -3\sin^2\theta + 3\cos^2\theta = 0$

$\frac{dr}{d\theta} = -3\sin\theta$

$-3\cdot\frac{1}{2}(1-\cos 2\theta) + 3\cdot\frac{1}{2}(1+\cos 2\theta) = 0$

$3\cos 2\theta = 0 \quad 2\theta = \pi/2 \text{ or } 2\theta = 3\pi/2$

$\Rightarrow \theta = \pi/4 \text{ or } \theta = 3\pi/4$

$r = 3\cos\pi/4 = 3\cdot\frac{\sqrt{2}}{2}$

$r = 3\cos 3\pi/4 = 3\cdot\left(\frac{-\sqrt{2}}{2}\right)$

$\Rightarrow$ The tangents are horizontal for $\left(\frac{3}{2}\sqrt{2}, \pi/4\right)$ and $\left(-\frac{3}{2}\sqrt{2}, 3\pi/4\right)$

Vertical: $\frac{dx}{d\theta} = 0$

$x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

$= (-3\sin\theta)\cos\theta - (3\cos\theta)\sin\theta$

$= -6\sin\theta\cos\theta = -6\cdot\frac{1}{2}\sin 2\theta$

$= -3\sin 2\theta$

$\frac{dx}{d\theta} = 0$ when $-3\sin 2\theta = 0$

$2\theta = 0$ and $2\theta = \pi$

$\theta = 0$ $\quad \theta = \pi/2$

$r = 3\cos 0 = 3$

$r = 3\cos \pi/2 = 0$

Points are $(3, 0)$ and $(0, \pi/2)$

5a) $r = 2\cos\theta$, $\theta = \pi/3$

slope of tangent line

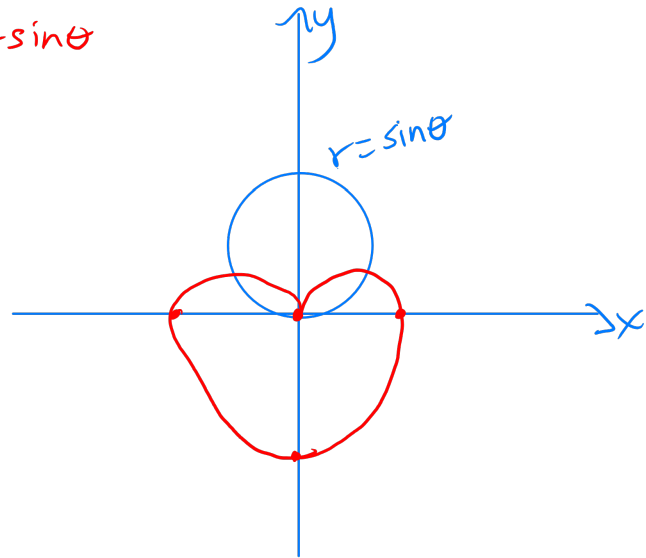
$m = \frac{dy}{dx}\Big|_{\theta=\pi/3}$ = homework

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{(-2\sin\theta)\sin\theta + (2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (2\cos\theta)\sin\theta}$$

10.4 5a) $r = \sin\theta$, $r = 1 - \sin\theta$

radius $1/2$ centered $(0, 1/2)$

θ	$r = 1 - \sin\theta$
0	1
$\pi/2$	0
π	1
$3\pi/2$	2



$\sin\theta = 1 - \sin\theta$

$2\sin\theta = 1$

$\sin\theta = 1/2$ $\Rightarrow \theta = \pi/6, 5\pi/6$

$r = \sin\pi/6 = 1/2$

$r = \sin 5\pi/6 = 1/2$ $\Rightarrow$ Points of intersection are $(\frac{1}{2}, \pi/6), (1/2, 5\pi/6), (0, 0)$

10.4 6) $r = 2\cos\theta, \ 0 \le \theta \le \pi$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$= \int_0^\pi \sqrt{4}\, d\theta = 2\theta \Big]_0^\pi = \boxed{2\pi}$$

Homework B.1: Three-Dimensional Coordinate Systems

Prof. Leyda Almodóvar

Name: _____

1. (a) What does the equation $y = 4$ represent in $\mathbb{R}^3$?

   In $\mathbb{R}^3$, the equation $y = 4$ represents a vertical plane that is parallel to the xz-plane and 4 units to the right of it.

   (b) What does $z = 6$ represent in $\mathbb{R}^3$?

   The equation $z = 6$ represents a horizontal plane parallel to the xy-plane and 6 units above it.

   (c) What does the pair of equations $y = 4$, $z = 6$ represent? In other words, describe the set of point (x, y, z) such that $y = 4$, $z = 6$.

   The pair of equations $y = 4$, $z = 6$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y = 4$, $z = 6$. The points of these line are of the form $(x, 4, 6)$, that is, the x-coordinate may vary but the y- and z-coordinates are fixed at 4 and 6, respectively.

2. Find the distance from $(4, -2, 6)$ to each of the following:

   (a) The xy-plane

   The distance from a point to the xy-plane is the absolute value of the z-coordinate of the point. Thus, the distance is $|6| = 6$.

   (b) The yz-plane

   The distance to the yz-plane is the absolute value of the x-coordinate of the point which is $|4| = 4$.

   (c) The xz-plane

   The distance to the xz-plane is the absolute value of the y-coordinate of the point which is $|-2| = 2$.

(d) The x-axis

The point on the x-axis closest to $(4, -2, 6)$ is the point $(4, 0, 0)$ (approaching the x-axis perpendicularly). The distance from $(4, -2, 6)$ to the x-axis is $\sqrt{(4-4)^2 + (-2-0)^2 + (6-0)^2} = \sqrt{40} = \boxed{2\sqrt{10}}$

(e) The y-axis

The point on the y-axis closest to $(4, -2, 6)$ is $(0, -2, 0)$. The distance between these points is $\sqrt{(4-0)^2 + [-2-(-2)]^2 + (6-0)^2} = \sqrt{52} = \boxed{2\sqrt{13}}$

(f) The z-axis

The point on the z-axis closest to $(4, -2, 6)$ is $(0, 0, 6)$. The distance between these points is $\sqrt{(4-0)^2 + (-2-0)^2 + (6-6)^2} = \sqrt{20} = \boxed{2\sqrt{5}}$

3. Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

An equation of the sphere with center $(2, -6, 4)$ and radius 5 is $(x-2)^2 + [y-(-6)]^2 + (z-4)^2 = 5^2$ or $(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$.

The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z = 0$ into the equation we have $(x-2)^2 + (y+6)^2 = 9$, $z = 0$ which represents a circle in the xy-plane with center $(2, -6, 0)$ and radius 3.

To find the intersection with the xz-plane, we set $y = 0$: $(x-2)^2 + (z-4)^2 = -11$. Since no points satisfy this equation, the sphere does not intersect the xz-plane.

To find the intersection with the yz-plane, we set $x = 0$: $(y+6)^2 + (z-4)^2 = 21$, $x = 0$, a circle in the yz-plane with center $(0, -6, 4)$ and radius $\sqrt{21}$.

4. Find an equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1, 2, 3)$: $r = \sqrt{(1-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{14}$. Then an equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$.

5. Describe in words the region of $\mathbb{R}^3$ represented by the equation(s) or inequality.

(a) $0 \le z \le 6$

The inequality $0 \le z \le 6$ represents all points on or between the horizontal planes $z = 0$ (the xy-plane) and $z = 6$.

(b) $x^2 + y^2 = 4$

Note there is no restriction on z, so a point in the region must lie on a circle of radius 2, center on the z-axis, but it could be in any horizontal plane $z = k$ (parallel to the xy-plane). Thus, the region consists of all possible circles $x^2 + y^2 = 4$, $z = k$ and is therefore a circular cylinder with radius 2 whose axis is the z-axis.

(c) $1 \le x^2 + y^2 + z^2 \le 5$

The inequalities $1 \le x^2 + y^2 + z^2 \le 5$ are equivalent to $1 \le \sqrt{x^2 + y^2 + z^2} \le \sqrt{5}$, so the region consists of those points whose distance from the origin is at least 1 and at most $\sqrt{5}$. This is the set of all points on or between spheres with radii 1 and $\sqrt{5}$ and centers $(0, 0, 0)$.

(d) $x^2 + z^2 \le 9$

Here $x^2 + z^2 \le 9$ or equivalently $\sqrt{x^2 + z^2} \le 3$ which describes the set of all points in $\mathbb{R}^3$ whose distance from the y-axis is at most 3. Thus the inequality represents the region consisting of all points on or inside a circular cylinder of radius 3 with axis the y-axis.

Homework B.2: Vectors

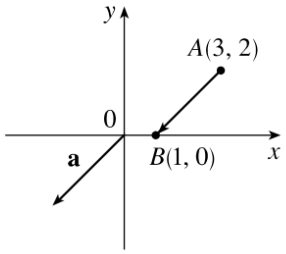
Prof. Leyda Almodóvar

Name: _____

1. Find a vector $\vec{a}$ with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

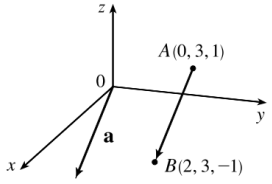
   (a) $A = (3, 2)$, $B = (1, 0)$

   $\vec{a} = \langle 1 - 3, 0 - 2 \rangle = \langle -2, -2 \rangle$



   (b) $A = (0, 3, 1)$, $B = (2, 3, -1)$

   $\vec{a} = \langle 2 - 0, 3 - 3, -1 - 1 \rangle = \langle 2, 0, -2 \rangle$



2. Find $\vec{a} + \vec{b}$, $4\vec{a} + 2\vec{b}$, $|\vec{a}|$, and $|\vec{a} - \vec{b}|$

   (a) $\vec{a} = 5\vec{i} + 3\vec{j}$, $\vec{b} = -\vec{i} - 2\vec{j}$

   $\vec{a} + \vec{b} = (5\vec{i} + 3\vec{j}) + (-\vec{i} - 2\vec{j}) = \boxed{4\vec{i} + \vec{j}}$

$4\vec{a} + 2\vec{b} = 4(5\vec{i} + 3\vec{j}) + 2(-\vec{i} - 2\vec{j}) = 20\vec{i} + 12\vec{j} - 2\vec{i} - 4\vec{j} = \boxed{18\vec{i} + 8\vec{j}}$

$|\vec{a}| = \sqrt{5^2 + 3^2} = \boxed{\sqrt{34}}$

$|\vec{a} - \vec{b}| = |(5\vec{i} + 3\vec{j}) - (-\vec{i} - 2\vec{j})| = |6\vec{i} + 5\vec{j}| = \sqrt{6^2 + 5^2} = \boxed{\sqrt{61}}$

(b) $\vec{a} = \langle 8, 1, -4\rangle$, $\vec{b} = \langle 5, -2, 1\rangle$

$\vec{a} + \vec{b} = \langle 8 + 5, 1 + (-2), -4 + 1\rangle = \boxed{\langle 13, -1, -3\rangle}$

$4\vec{a} + 2\vec{b} = 4\langle 8, 1, -4\rangle + 2\langle 5, -2, 1\rangle = \langle 32, 4, -16\rangle + \langle 10, -4, 2\rangle = \langle 42, 0, -14\rangle$

$|\vec{a}| = \sqrt{8^2 + 1^2 + (-4)^2} = \sqrt{81} = \boxed{9}$

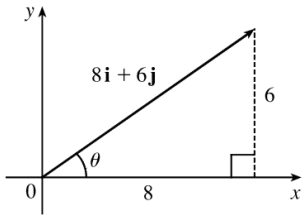
$|\vec{a} - \vec{b}| = |\langle 8 - 5, 1 - (-2), -4 - 1\rangle| = |\langle 3, 3, -5\rangle| = \sqrt{3^2 + 3^2 + (-5)^2} = \boxed{\sqrt{43}}$

3. Find a unit vector that has the same direction as $-5\vec{i} + 3\vec{j} - \vec{k}$.

The vector $-5\vec{i} + 3\vec{j} - \vec{k}$ has length $|-5\vec{i} + 3\vec{j} - \vec{k}| = \sqrt{(-5)^2 + 3^2 + (-1)^2} = \sqrt{35}$

The unit vector with the same direction is $\frac{1}{\sqrt{35}}(-5\vec{i} + 3\vec{j} - \vec{k}) = \boxed{-\frac{5}{\sqrt{35}}\vec{i} + \frac{3}{\sqrt{35}}\vec{j} - \frac{1}{\sqrt{35}}\vec{k}}$

4. What is the angle between the vector $8\vec{i} + 6\vec{j}$ and the positive direction of the x-axis?



From the figure we see that $\tan\theta = \frac{6}{8} = \frac{3}{4}$, so $\boxed{\theta = \tan^{-1}\left(\frac{3}{4}\right)}$

B.3 The Dot Product and Cross Product

Multiplication of two vectors is not *exactly* a thing like multiplication of two numbers, but there are other two operations on two vectors besides addition and subtraction, and they do involve some multiplication.

The Dot Product

Definition. The **dot product** of two vectors $\vec{v}$ and $\vec{w}$ is given by:

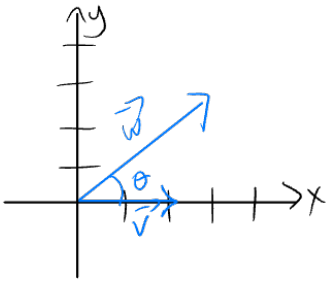
- $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, or
- $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$ (in radians)

$= \langle v_1, v_2, v_3 \rangle$, $= \langle w_1, w_2, w_3 \rangle$

For 2-D: • $\vec{v}\cdot\vec{w} = v_1 w_1 + v_2 w_2$ • $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$

Note. The dot product is a *number*.

Example. $\vec{v} = 2\vec{i}$, $\vec{w} = 3\vec{i} + 3\vec{j}$. Draw the vectors, then compute the dot product in *both* ways.



$\vec{v} = 2\vec{i} = \langle 2, 0 \rangle$, $\vec{w} = \langle 3, 3 \rangle$

$\vec{v} \cdot \vec{w} = 2 \cdot 3 + 0 \cdot 3 = \boxed{6}$

$|\vec{v}| = \sqrt{2^2 + 0^2} = 2$

$|\vec{w}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\theta = \pi/4$

$\vec{v} \cdot \vec{w} = 2 \cdot 3\sqrt{2} \cdot \cos\pi/4$

$= 6\sqrt{2} \cdot \frac{\sqrt{2}}{2} = \boxed{6}$

Properties of the Dot Product

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutative)
2. $\vec{v} \cdot (c\vec{w}) = c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w}$ (scalar commutativity)
3. $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$ (distributive)

True Fact. Two nonzero vectors $\vec{v}$ and $\vec{w}$ are perpendicular (or "orthogonal") if and only if $\vec{v} \cdot \vec{w} = 0$. Why? (Hint: Look at the definition with an angle value.)

• If $\vec{v}$ and $\vec{w}$ are perpendicular, $\theta = \pi/2$ and $\cos\pi/2 = 0 \Rightarrow \vec{v} \cdot \vec{w} = 0$

• If $\vec{v} \cdot \vec{w} = 0$ and $\vec{v} \neq \vec{0}$, $\vec{w} \neq \vec{0}$, then $\cos\theta = 0 \Rightarrow \theta = \pi/2 \Rightarrow \vec{v} \perp \vec{w}$

True Fact. $\vec{v} \cdot \vec{v} = |\vec{v}|^2$. Why?

$\vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 + v_3 v_3 = v_1^2 + v_2^2 + v_3^2$

$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Example. Compute the dot products of all possible combinations of the unit vectors $(\vec{i}, \vec{j}, \vec{k})$ together.

$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

$\vec{i} \cdot \vec{j} = 0$, $\vec{i} \cdot \vec{k} = 0$, $\vec{j} \cdot \vec{k} = 0$, $\vec{j} \cdot \vec{i} = 0$, $\vec{k} \cdot \vec{i} = 0$, $\vec{k} \cdot \vec{j} = 0$

$\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$, $\vec{k} \cdot \vec{k} = 1$

Example. Give a vector perpendicular to $\vec{v} = -3\vec{i} + 4\vec{k}$.

Let $\vec{w} = \langle w_1, w_2, w_3 \rangle$ be a vector such that $\vec{v} \cdot \vec{w} = 0$. Then $-3w_1 + 0 + 4w_3 = 0$

$\Rightarrow 4w_3 = 3w_1 \Rightarrow w_3 = \frac{3}{4}w_1$.

* w_2 can be anything and our choice of w_1 determines w_3

$w_1 = 8 \Rightarrow w_3 = \frac{3}{4} \cdot 8 = 6$ $\quad \boxed{\vec{w} = \langle 8, 1, 6 \rangle}$ $\quad$ * not unique!

Example. Find the angle between the vectors $\vec{a} = \langle 4, 3 \rangle$, $\vec{b} = \langle 2, -1 \rangle$.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\vec{a} \cdot \vec{b} = 4 \cdot 2 + 3 \cdot (-1) = 5$

$|\vec{a}| = \sqrt{4^2 + 3^2} = 5$

$|\vec{b}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$5 = 5\sqrt{5} \cos\theta$

$\frac{1}{\sqrt{5}} = \cos\theta$

$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}$

Example. Find a unit vector that is orthogonal to both $\vec{i} + \vec{j}$ and $\vec{i} + \vec{k}$.

Let $\vec{v}$ such that $|\vec{v}| = 1$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Also, $\vec{v} \cdot (\vec{i} + \vec{j}) = 0$ and $\vec{v} \cdot (\vec{i} + \vec{k}) = 0$

$\langle v_1, v_2, v_3 \rangle \cdot \langle 1, 1, 0 \rangle = 0 \Rightarrow v_1 + v_2 = 0 \Rightarrow v_2 = -v_1$

$\langle v_1, v_2, v_3 \rangle \cdot \langle 1, 0, 1 \rangle = 0 \Rightarrow v_1 + v_3 = 0 \Rightarrow v_3 = -v_1$

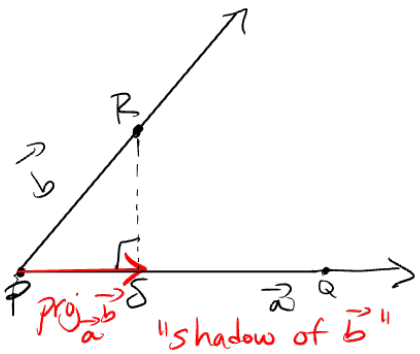
$1 = \sqrt{v_1^2 + (v_1)^2 + (v_1)^2} = \sqrt{3v_1^2} = \pm\sqrt{3}v_1 \Rightarrow v_1 = \pm\frac{1}{\sqrt{3}}$

$\vec{v} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$ or $\vec{v} = \left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

Projections

Definition. Let $\vec{PQ} = \vec{a}$ and $\vec{PR} = \vec{b}$ be two vectors with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\vec{PQ}$, then the vector with representation $\vec{PS}$ is called the **vector projection of** $\vec{b}$ **onto** $\vec{a}$ and is denoted by $\text{proj}_{\vec{a}}\vec{b}$.

Fact. To get the projection of $\vec{b}$ onto $\vec{a}$ we drop straight down from the end of $\vec{b}$ until we form a right angle with the line that is parallel to $\vec{a}$.



$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right) \cdot \vec{a} \neq \text{proj}_{\vec{b}}\vec{a}$

* This is a vector!

Example. Find the projection of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\cdot\vec{a} = \frac{3}{14}\cdot\langle -2,3,1\rangle = \boxed{\left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle}$$

$\vec{a}\cdot\vec{b} = -2\cdot 1 + 3\cdot 1 + 1\cdot 2 = 3$

$|\vec{a}| = \sqrt{(-2)^2+3^2+1^2} = \sqrt{14}$

Definition. The **scalar projection of $\vec{b}$ onto $\vec{a}$** is defined to be the signed magnitude of the vector projection, which is the number $|\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, and is denoted by $\text{comp}_{\vec{a}}\vec{b}$.

$$\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

Recall $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

* This is a number!

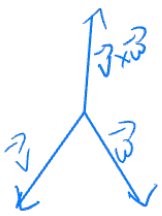
Example. Find the scalar projection of $\vec{b} = \langle 1, 1, 2\rangle$ onto $\vec{a} = \langle -2, 3, 1\rangle$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \boxed{\frac{3}{\sqrt{14}}}$$

The Cross Product

Definition. The **cross product** of two vectors $\vec{v}$ and $\vec{w}$, written $\vec{v}\times\vec{w}$, is a vector perpendicular to both $\vec{v}$ and $\vec{w}$.

- (Geometric definition) $\vec{v}\times\vec{w} = (|\vec{v}||\vec{w}|\sin\theta)\vec{n}$ where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$, $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ in the direction given by the right-hand rule.
  - Sketch:



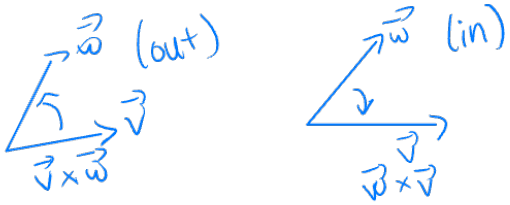
- (Algebraic definition) $\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$.
  - Determinant explanation for this formula:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$
$$= (v_2w_3 - v_3w_2)\vec{i} - (v_1w_3 - v_3w_1)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

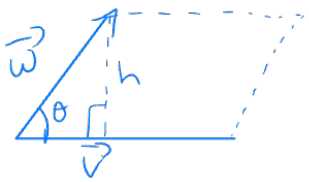
Note. The cross product is a *vector*.

The direction of the cross product $\vec{v} \times \vec{w}$ is given by the right hand rule:

- To determine whether the cross product vector points in or out: With your right hand, place your fingers along $\vec{v}$ so that you can easily "squash" $\vec{v}$ into $\vec{w}$. Your thumb points along $\vec{n}$. (Note: This means the order matters.)
  Sketch:



Fact. The magnitude of the cross product is the area of the parallelogram formed by the two vectors.



$$A = |\vec{v}||\vec{w}|\sin\theta = |\vec{v} \times \vec{w}|$$

Note. The cross product is only defined for 3D vectors.

Note. The cross product is **not** commutative ($\vec{v} \times \vec{w} \neq \vec{w} \times \vec{v}$).

Algebraic Facts:

$\theta = 0$ and $\sin 0 = 0$

- If $\vec{v}$ and $\vec{w}$ are parallel, $\vec{v} \times \vec{w} = \vec{0}$. In particular, $\vec{v} \times \vec{v} = \vec{0}$ since a vector is parallel to itself.
- $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$ (Why?)

$$\vec{w} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} w_2 & w_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} w_1 & w_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} w_1 & w_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$
$$= (w_2v_3 - v_2w_3)\vec{i} - (w_1v_3 - v_1w_3)\vec{j} + (w_1v_2 - v_1w_2)\vec{k}$$

$$\vec{v} \times \vec{w} = (v_2w_3 - w_2v_3)\vec{i} - (v_1w_3 - w_1v_3)\vec{j} + (v_1w_2 - w_1v_2)\vec{k}$$

- $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w}) = \vec{v} \times (c\vec{w})$

  We can distribute scalars across a cross product

- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

  Distributivity

Example. Compute all cross product combinations of $\vec{i}, \vec{j}, \vec{k}$.

$\vec{i} = \langle 1,0,0 \rangle$
$\vec{j} = \langle 0,1,0 \rangle$
$\vec{k} = \langle 0,0,1 \rangle$

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$

$\vec{i} \times \vec{j} = \langle 0-0, 0-0, 1-0 \rangle = \vec{k}$

$\vec{j} \times \vec{i} = -(\vec{i} \times \vec{j}) = -\vec{k}$

$\vec{i} \times \vec{k} = \langle 0-0, 0-0, 1 \cdot 0 - 0 \cdot 1 \rangle = -\vec{j}$

$\vec{k} \times \vec{i} = \vec{j}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{j} = -\vec{i}$

Example. Compute the cross product of $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$ and verify that your resulting vector is perpendicular to both $\vec{v}$ and $\vec{w}$.

$\vec{v} = \langle 2, -3, 1 \rangle$, $\vec{w} = \langle 1, 2, -1 \rangle$

$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}$

or

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}\vec{k}$$

$= (3-2)\vec{i} - (-2-1)\vec{j} + (4+3)\vec{k} = \vec{i} + 3\vec{j} + 7\vec{k}$

$= \langle 1, 3, 7 \rangle$

$(\vec{v} \times \vec{w}) \cdot \vec{v} = \langle 1,3,7 \rangle \cdot \langle 2, -3, 1 \rangle = 1 \cdot 2 + 3 \cdot (-3) + 7 \cdot 1 = 0$

$(\vec{v} \times \vec{w}) \cdot \vec{w} = \langle 1,3,7 \rangle \cdot \langle 1, 2, -1 \rangle = 1 \cdot 1 + 3 \cdot 2 + 7 \cdot (-1) = 0$

$\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

Homework B.3: The Dot Product and Cross Product

Prof. Leyda Almodóvar

Name: _____

1. Find $\vec{a} \cdot \vec{b}$.

   (a) $\vec{a} = \langle 6, -2, 3 \rangle$, $\vec{b} = \langle 2, 5, -1 \rangle$

   $\vec{a} \cdot \vec{b} = \langle 6, -2, 3 \rangle \cdot \langle 2, 5, -1 \rangle = (6)(2) + (-2)(5) + (3)(-1) = 12 - 10 - 3 = \boxed{-1}$

   (b) $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$, $\vec{b} = 4\vec{i} + 5\vec{k}$

   $\vec{a} \cdot \vec{b} = (3\vec{i} + 2\vec{j} - \vec{k}) \cdot (4\vec{i} + 5\vec{k}) = (3)(4) + (2)(0) + (-1)(5) = \boxed{7}$

   (c) $|\vec{a}| = 80$, $|\vec{b}| = 50$, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{3\pi}{4}$

   We know $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = (80)(50)\cos\frac{3\pi}{4} = 4000\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-2000\sqrt{2}}$

2. Find the angle between the vectors. (Find an exact expression, not an approximation.)

   (a) $\vec{a} = \langle -2, 5 \rangle$, $\vec{b} = \langle 5, 12 \rangle$

   $|\vec{a}| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$, $|\vec{b}| = \sqrt{5^2 + 12^2} = 13$, $\vec{a} \cdot \vec{b} = (-2)(5) + (5)(12) = 50$.

   Then, $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{50}{\sqrt{29} \cdot 13} = \frac{50}{13\sqrt{29}}$ and the angle between $\vec{a}$ and $\vec{b}$ is $\boxed{\theta = \cos^{-1}\left(\frac{50}{13\sqrt{29}}\right)}$

   (b) $\vec{a} = \langle -1, 3, 4 \rangle$, $\vec{b} = \langle 5, 2, 1 \rangle$

   $|\vec{a}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}$, $|\vec{b}| = \sqrt{5^2 + 2^2 + 1^2} = \sqrt{30}$,

   $\vec{a} \cdot \vec{b} = (-1)(5) + (3)(2) + (4)(1) = 5$.

   Then, $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{26} \cdot \sqrt{30}} = \frac{5}{\sqrt{780}} = \frac{5}{2\sqrt{195}}$ and $\boxed{\theta = \cos^{-1}\left(\frac{5}{2\sqrt{195}}\right)}$

(c) $\vec{a} = 4\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{k}$

$|\vec{a}| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{26}$, $|\vec{b}| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5}$

$\vec{a} \cdot \vec{b} = (4)(2) + (-3)(0) + (1)(-1) = 7.$

Then, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{7}{\sqrt{26}\cdot\sqrt{5}} = \frac{7}{\sqrt{130}}$ and $\boxed{\theta = \cos^{-1}\left(\frac{7}{\sqrt{130}}\right)}$

3. Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\vec{u} = \langle -5, 4, -2\rangle$, $\vec{v} = \langle 3, 4, -1\rangle$

$\vec{u} \cdot \vec{v} = (-5)(3) + (4)(4) + (-2)(-1) = 3 \neq 0$, so $\vec{u}$ and $\vec{v}$ are not orthogonal. Also, $\vec{u}$ is not a scalar multiple of $\vec{v}$, so $\vec{u}$ and $\vec{v}$ are not parallel.

(b) $\vec{u} = 9\vec{i} - 6\vec{j} + 3\vec{k}$, $\vec{v} = -6\vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{u} \cdot \vec{v} = (-9)(-6) + (-6)(4) + (3)(-2) = -84 \neq 0$, so $\vec{u}$ and $\vec{v}$ are not orthogonal. Because $\vec{u} = -\frac{3}{2}\vec{v}$, $\vec{u}$ and $\vec{v}$ are parallel.

(c) $\vec{u} = \langle c, c, c, \rangle$, $\vec{v} = \langle c, 0, -c\rangle$

$\vec{u} \cdot \vec{v} = (c)(c) + (c)(0) + (c)(-c) = c^2 + 0 - c^2 = 0$, $\vec{u}$ and $\vec{v}$ are orthogonal (and not parallel). Note that if $c = 0$, $\vec{u} = \vec{v} = \vec{0}$, and the zero vector is considered orthogonal to all vectors. Although in this case $\vec{u}$ and $\vec{v}$ are identical, they are not considered parallel, as only nonzero vectors can be parallel.

4. Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$ where $\vec{a} = \langle -1, 4, 8\rangle$, $\vec{b} = \langle 12, 1, 2\rangle$

$|\vec{a}| = \sqrt{(-1)^2 + 4^2 + 8^2} = \sqrt{81} = 9,$

the scalar projection of $\vec{b}$ onto $\vec{a}$ is $\text{comp}_a\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{(-1)(12)+(4)(1)+(8)(2)}{2} = \boxed{\frac{8}{9}}$

The vector projection of $\vec{b}$ onto $\vec{a}$ is $\text{proj}_a\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{a}{|\vec{a}|} = \frac{8}{9}\cdot\frac{1}{9}\langle -1, 4, 8\rangle = \frac{8}{81}\langle -1, 4, 8\rangle =$

$\boxed{\left\langle -\frac{8}{81}, \frac{32}{81}, -\frac{64}{81}\right\rangle}$

5. Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = 3\vec{i} + 3\vec{j} - 3\vec{k}$, $\vec{b} = 3\vec{i} - 3\vec{j} + 3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 3 & -3 \\ 3 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -3 \\ -3 & 3 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & -3 \\ 3 & 3 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 3 \\ 3 & -3 \end{vmatrix}\vec{k}$$

$= (9-9)\vec{i} - [9-(-9)]\vec{j} + (-9-9)\vec{k} = -18\vec{j} - 18\vec{k}$

Since $(\vec{a} \times \vec{b}) \cdot \vec{a} = (-18\vec{j} - 18\vec{k}) \cdot (3\vec{i} + 3\vec{j} - 3\vec{k}) = 0 - 54 + 54 = 0$, $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$.

Since $(\vec{a} \times \vec{b}) \cdot \vec{b} = (-18\vec{j} - 18\vec{k}) \cdot (3\vec{i} - 3\vec{j} + 3\vec{k}) = 0 + 54 - 54 = 0$, $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$.

(b) $\vec{a} = t\vec{i} + \cos t\vec{j} + \sin t\vec{k}$, $\vec{b} = \vec{i} - \sin t\vec{j} + \cos t\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}\vec{i} - \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix}\vec{j} + \begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix}\vec{k}$$

$= [\cos^2 t - (-\sin^2 t)]\vec{i} - (t\cos t - \sin t)\vec{j} + (-t\sin t - \cos t)\vec{k}$

$= \vec{i} + (\sin t - t\cos t)\vec{j} + (-t\sin t - \cos t)\vec{k}$

Since $(\vec{a} \times \vec{b}) \cdot \vec{a} = [\vec{i} + (\sin t - t\cos t)\vec{j} + (-t\sin t - \cos t)\vec{k}] \cdot (t\vec{i} + \cos t\vec{j} + \sin t\vec{k})$

$= t + \sin t\cos t - t\cos^2 t - t\sin^2 t - \sin t\cos t = t - t(\cos^2 t + \sin^2 t) = 0,$

$\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$.

Since $(\vec{a} \times \vec{b}) \cdot \vec{b} = [\vec{i} + (\sin t - t\cos t)\vec{j} + (-t\sin t - \cos t)\vec{k}] \cdot (\vec{i} - \sin t\vec{j} + \cos t\vec{k})$

$= 1 - \sin^2 t + t\sin t\cos t - t\sin t\cos t - \cos^2 t = 1 = (\sin^2 t + \cos^2 t) = 0,$

$\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$.

6. If $\vec{a} = \langle 2, -1, 3\rangle$, and $\vec{b} = \langle 4, 2, 1\rangle$, find $a \times b$ and $b \times a$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} \vec{k}$$

$$= (-1-6)\vec{i} - (2-12)\vec{j} + [4-) - 4)]\vec{k} = -7\vec{i} + 10\vec{j} + 8\vec{k}$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} \vec{k}$$

$$= (-1-6)\vec{i} - (2-12)\vec{j} + [4-) - 4)]\vec{k} = 7\vec{i} - 10\vec{j} - 8\vec{k}$$

7. Find two unit vectors orthogonal to both $\langle 3, 2, 1\rangle$ and $\langle -1, 1, 0\rangle$.

The cross product of two vectors is orthogonal to both vectors. So we calculate $\langle 3, 2, 1\rangle \times \langle -1, 1, 0\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} \vec{k} = -\vec{i} - \vec{j}_5[\vec{k}$

So two unit vectors orthogonal to both given vectors are $\pm\frac{\langle -1,-1,5\rangle}{\sqrt{1+1+25}} = \pm\frac{\langle -1,-1,5\rangle}{3\sqrt{3}}$, that is, $\left\langle -\frac{1}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}\right\rangle$ and $\left\langle \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, -\frac{5}{3\sqrt{3}}\right\rangle$.

8. Given $P = (0, 0, -3)$, $Q = (4, 2, 0)$, and $R = (3, 3, 1)$,

   (a) find a nonzero vector orthogonal to the plane through the points P, Q, and R, and

   $\overrightarrow{PQ} = \langle 4, 3, 2\rangle$ and $\overrightarrow{PR} = \langle 3, 3, 4\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(4) - (3)(3), (3)(3) - (4)(4), (4)(3) - (2)(3)\rangle = \langle -1, -7, 6\rangle$ (or any nonzero scalar multiple thereof).

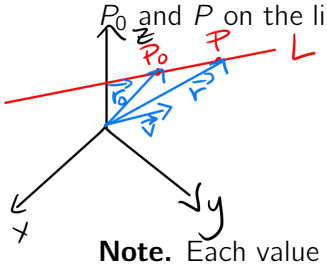
   (b) find the area of triangle PQR.

   The area of the parallelogram determine by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -1, -7, 6\rangle| = \sqrt{1 + 49 + 36} = \sqrt{86}$, so the area of the triangle PQR is $\frac{1}{2}\sqrt{86}$.

B.4 Normal Vectors & Equations of Lines and Planes

In three-dimensional space, a line L is determined when we know a point $P_0 = (x_0, y_0, z_0)$ on L and the direction of L. The direction of a line is conveniently described by a vector.

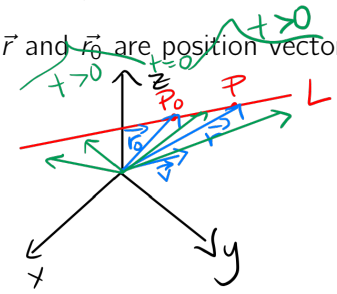
Definition. The **vector equation** of L is $\vec{r} = \vec{r_0} + t\vec{v}$, where $\vec{r}$ and $\vec{r_0}$ are position vectors of points P_0 and P on the line.



$\vec{v}$ is a vector parallel to L

$P_0 = (x_0, y_0, z_0)$

$P = (x, y, z)$



Note. Each value of the parameter t gives the position vector $\vec{r}$ of a point on L.

Note. As t varies, the line is traced out by the top of the vector $\vec{r}$.

Note. $t > 0$ correspond to points that lie on one side of P_0, while $t < 0$ correspond to points that lie on the other side of P_0.

Fact. Two vectors are equal if and only if their corresponding components are equal.

Since $\vec{r} = \vec{r_0} + t\vec{v}$, $\vec{v} = \langle a, b, c\rangle$, $\vec{r} = \langle x, y, z\rangle$ and $\vec{r_0} = \langle x_0, y_0, z_0\rangle$, we obtain:

$\vec{r} = \vec{r_0} + t\vec{v} \Rightarrow \langle x, y, z\rangle = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$

Definition. The **parametric equations** for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are

$$x = x_0 + at, \qquad y = y_0 + bt \qquad z = z_0 + ct$$

Example. Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$. Then find two other points on the line.

Vector equation: $\vec{r} = \vec{r_0} + t\vec{v} = \langle 5, 1, 3\rangle + t\langle 1, 4, -2\rangle$

$= \langle 5, 1, 3\rangle + \langle t, 4t, -2t\rangle$

$= \boxed{\langle 5 + t, 1 + 4t, 3 - 2t\rangle}$

Parametric equations:

$$\boxed{\begin{aligned} x &= 5 + t \\ y &= 1 + 4t \\ z &= 3 - 2t \end{aligned}}$$

$t = 1$: $x = 5 + 1 = 6$, $y = 1 + 4(1) = 5$, $z = 3 - 2(1) = 1$ $\Rightarrow (6, 5, 1)$ is a point on the line

$t = 4$: $x = 5 + 4 = 9$, $y = 1 + 4(4) = 17$, $z = 3 - 2(4) = -5$, $(9, 17, -5)$ is a point on the line

t can be negative!

Example. Determine whether the lines L_1: $x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$ and L_2: $x = 1 + 4s$, $y = 3 - 2s$, $z = 4 + 5s$ are parallel, skew, or intersecting.

not parallel but don't intersect either

*Find the direction vectors of L_1 and L_2: L_1: $\langle 2, -1, 3 \rangle$

L_2: $\langle 4, -2, 5 \rangle$

$\frac{4}{2} = 2 = \frac{-2}{-1} \neq \frac{5}{3}$ $\Rightarrow$ not scalar multiples $\Rightarrow$ not parallel

If the lines intersect, we must have $3+2t = 1+4s$, $4-t = 3-2s$, $1+3t = 4+5s$

times 3

$12 - 3t = 9 - 4s$

$+ \quad 1 + 3t = 4 + 5s$

$13 = 13 + s \Rightarrow s = 0$

Substitute into all equations. If we obtain the same value of t for all, the lines intersect. Otherwise these are skew.

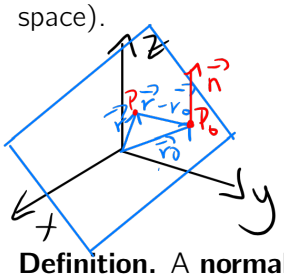
$3 + 2t = 1 + 4(0) \Rightarrow t = -1$

$4 - t = 3 - 2(0) \Rightarrow t = 1$

$\Rightarrow$ SKEW!

In two variables, a linear function is one whose graph is a **plane** (like an infinite rectangle in 3-space).

$f(x, y) = mx + ny + c$



Definition. A **normal vector** to a plane is a vector that is perpendicular to the plane.

- This means the normal vector is perpendicular (dot product 0) to any displacement vector between any two points in the plane.

Planes are fully defined by their normal vectors:

Definition. The **equation of the plane** with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P_0 = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

If we set $d = ax_0 + by_0 + cz_0$ (note this is a constant) then $ax + by + cz = d$.

Note. a/c is the slope in the x-direction (as you move along the x-axis; change in z over change in x) and b/c is the slope in the y-direction (as you move along the y-axis; change in z over change in y).

Thus, a linear function (plane) with slope m in the x direction and slope n in the y direction has the form

$$f(x, y) = mx + ny + c$$

More specifically, if f passes through (x_0, y_0, z_0), then it has the form

$$f(x, y) = m(x - x_0) + n(y - y_0) + z_0$$

Example. Find the equation of the plane perpendicular to $2\vec{i} - 3\vec{j} + 7\vec{k}$ and passing through the point $(1, -1, 2)$.

$a = 2$, $b = -3$, $c = 7$; $x_0 = 1$, $y_0 = -1$, $z_0 = 2$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$
$$2(x-1)-3(y+1)+7(z-2)=0$$
$$2x-2-3y-3+7z-14=0$$
$$\boxed{2x-3y+7z=19}$$

Example. Find a normal vector to the plane with the equation $z = 2x + 3y$.

$$2x+3y-z=0$$

comparing to $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ we obtain $\boxed{\vec{n}=\langle 2,3,-1\rangle}$

Example. Find the point at which the line $x = 2 - 2t$, $y = 3t$, $z = 1 + t$ intersects the plane $x + 2y - z = 7$.

$$2-2t+2(3t)-(1+t)=7$$
$$2-2t+6t-1-t=7$$
$$3t+1=7$$
$$3t=6$$
$$t=2$$

$x = 2-2(2) = -2$
$y = 3(2) = 6$
$z = 1+2 = 3$

$\Rightarrow \boxed{(-2, 6, 3)}$

Lab 3: Normal Vectors & Plane Equations

In this lab you will explore graphs of planes through given points.

1. Open up a 3D graphing page on *Geogebra* and graph the points $P = (0, 1, 1), Q = (1, 0, 1), R = (1, 1, 0)$.

2. Compute the displacement vectors $\vec{PQ}$ and $\vec{PR}$.

$\vec{PQ} = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$

$\vec{PR} = \langle 1-0, 1-1, 0-1 \rangle = \langle 1, 0, -1 \rangle$

3. Add the vectors $\vec{PQ}$ and $\vec{PR}$ to your Geogebra graph. Graph these vectors so that they begin and end at one of P, Q, R instead of starting at the origin. Vector(P,Q) Vector(P,R)

4. Compute the cross product $\vec{PQ} \times \vec{PR}$.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}\vec{k}$$

$$= (1-0)\vec{i} - (-1-0)\vec{j} + (0+1)\vec{k}$$

$$= \vec{i} + \vec{j} + \vec{k} = \langle 1, 1, 1 \rangle$$

5. Add your resulting cross product vector to your Geogebra graph. Graph the vector with initial point P. Cross(u,v)

6. Write the equation of the plane through P, Q, R using the cross product vector that you found as the normal vector and the point P as the point P_0.

$\vec{n} = \langle 1, 1, 1 \rangle$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$1(x-1) + 1(y-1) + 1 \cdot (z-1) = 0$

$$\boxed{x + y + z = 2}$$

7. Graph your plane equation in Geogebra. Plane(P,Q,R)

8. Does the plane you've graphed contain all three points P, Q, R? What do you notice about the cross product (normal) vector and the plane?

Yes!

It was perpendicular to the plane.

Lab 3 Follow-Up Notes

In the lab we acquired a new method for finding an equation of a plane when we are given three points in the plane but *not* given a normal vector. We have discovered that a cross product of two vectors in the plane *is* a normal vector to the plane.

Suppose we are given three points P, Q, R. Then we can:

1. Calculate the displacement vectors $\vec{PQ}$ and $\vec{PR}$ (note that both vectors lie in the plane). same initial point
2. Calculate the cross product $\vec{PQ} \times \vec{PR}$ (note this is perpendicular to the plane and thus is a normal vector).
3. Use $\vec{PQ} \times \vec{PR}$ for $\vec{n}$ and any one of the points P, Q, R for P_0.

Example. Find the equation for the plane containing the points $(3, 4, 2), (-2, 1, 0), (0, 2, 1)$. P Q R

$\vec{PQ} = \langle -2-3, 1-4, 0-2 \rangle = \langle -5, -3, -2 \rangle$

$\vec{PR} = \langle 0-3, 2-4, 1-2 \rangle = \langle -3, -2, -1 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -3 & -2 \\ -3 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ -2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} -5 & -2 \\ -3 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} -5 & -3 \\ -3 & -2 \end{vmatrix} \vec{k}$$

$= (3-4)\vec{i} - (5-6)\vec{j} + (10-9)\vec{k} = -\vec{i} + \vec{j} + \vec{k}$ ← normal vector

$= \langle -1, 1, 1 \rangle$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$R = P_0$

$-1(x-0) + 1(y-2) + 1(z-1) = 0$

$-x + y - 2 + z - 1 = 0$

$$\boxed{-x + y + z = 3}$$

$Q = P_0$: $-1(x+2) + 1(y-1) + 1(z-0) = 0$

$-x - 2 + y - 1 + z = 0$

$$\boxed{-x + y + z = 3}$$

Homework B.4: Normal Vectors & Equations of Lines and Planes

Prof. Leyda Almodóvar

Name: _____

1. Find a vector equation and parametric equations for the line.

   (a) The line through the point $(6, -5, 2)$ and parallel to the vector $\langle 1, 3, -\frac{2}{3}\rangle$

   For this line we have $\vec{r_0} = 6\vec{i} - 5\vec{j} + 2\vec{k}$ and $\vec{v} = \vec{i} + 3\vec{j} - \frac{2}{3}\vec{k}$, so a vector equation is $\vec{r} = \vec{r_0} + t\vec{v} = (6\vec{i} - 5\vec{j} + 2\vec{k}) + t(\vec{i} + 3\vec{j} - \frac{2}{3}\vec{k}) = (6+t)\vec{i} + (-5+3t)\vec{j} + (2 - \frac{2}{3}t)\vec{k}$ and parametric equations are $x = 6 + t$, $y = -5 + 3t$, $z = 2 - \frac{2}{3}t$.

   (b) The line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$.

   This line has the same direction as the given line, $\vec{v} = 2\vec{i} - 3\vec{j} + 9\vec{k}$. Here $r_0 = 14\vec{j} - 10\vec{k}$, so a vector equation is $\vec{r} = (14\vec{j} - 10\vec{k}) + t(2\vec{i} - 3\vec{j} + 9\vec{k}) = 2t\vec{i} + (14 - 3t)\vec{j} + (-10 + 9t)\vec{k}$ and parametric equations are $x = 2t$, $y = 14 - 3t$, $z = -10 + 9t$.

2. Find parametric equations for the line through $(2, 1, 0)$ and perpendicular to both $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$.

   $\vec{v} = (\vec{i} + \vec{j}) \times (\vec{j} + \vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i} - \vec{j} + \vec{k}$ is the direction of the line perpendicular to both $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$.

   With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t$, $y = 1 - t$, $z = t$.

3. Find parametric equations for the line through $(2, 4, 6)$ that is perpendicular to the plane $x - y + 3z = 7$.

   A vector normal to the plane $x - y + 3z = 7$ is $\vec{n} = \langle 1, -1, 3\rangle$, and since the line is to be perpendicular to the plane, $\vec{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 2 + t$, $4 - t$, $z = 6 + 3t$.

4. Determine whether the lines L_1: $x = 5-12t$, $y = 3+9t$, $z = 1-3t$ and L_2: $x = 3+8s$, $y = -6s$, $z = 7+2s$ are parallel, skew, or intersecting. If they intersect, find the point of intersection.

   The direction vectors are $\vec{v_1} = \langle -12, 9, -3\rangle$ and $\vec{v_2} = \langle 8, -6, 2\rangle$, we have $\vec{v_1} = -\frac{3}{2}\vec{v_2}$ so the lines are parallel.

5. Find an equation of the plane through the point $(5, 3, 5)$ and with normal vector $2\vec{i} + \vec{j} - \vec{k}$.

   $2\vec{i} + \vec{j} - \vec{k} = \langle 2, 1, -1\rangle$ is a normal vector to the plane and $(5, 3, 5)$ is a point on the plane, so setting $a = 2$, $b = 1$, $c = -1$, $x_0 = 5$, $y_0 = 3$, $z_0 = 5$ gives
   $2(x-5) + 1(y-3) + (-1)(z-5) = 0$ or $2x + y - z = 8$ as an equation of the plane.

6. Find an equation of the plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$.

   Since the line is perpendicular to the plane, its direction vector $\langle 3, -1, 4\rangle$ is a normal vector to the plane. The point $(2, 0, 1)$ is on the plane, so an equation of the plane is $3(x-2) + (-1)(y-0) + 4(z-1) = 0$ or $3x - y + 4z = 10$.

7. Find an equation of the plane through the origin and the points $(3, -2, 1)$ and $(1, 1, 1)$.

   Here the vectors $\vec{a} = \langle 3, -2, 1\rangle$ and $\vec{b} = \langle 1, 1, 1, \rangle$ lie in the plane, so $\vec{n} = \vec{a} \times \vec{b} = \langle (-2)(1) - (1)(1), (1)(1) - (3)(1) - (-2)(1)\rangle = \langle -3, -2, 5\rangle$ is a normal vector to the plane. We can take the origin as P_0, so an equation of the plane is
   $-3(x-0) - 2(y-0) + 5(z-0) = 0$ or $-3x - 2y + 5z = 0$.

8. Find an equation of the plane through the points $(0, -2, 5)$ and $(-1, 3, 1)$ which is perpendicular to the plane $2z = 5x + 4y$.

   The points $(0, -2, 5)$ and $(-1, 3, 1)$ lie on the desired plane, so the vector $\vec{v_1} = \langle -1, 5, -4\rangle$ connecting them is parallel to the plane. The desired plane is perpendicular to the plane $2z = 5x + 4y$ or $5x + 4y - 2 = 0$ and for perpendicular planes, a normal vector for one plane is parallel to the other plane, so $\vec{v_2} = \langle 5, 4, -2\rangle$ is also parallel to the desired plane. A normal vector to the desired plane is
   $\vec{n} = \vec{v_1} \times \vec{v_2} = \langle -10 + 16, -20 - 2, -4 - 25\rangle = \langle 6, -22, -29\rangle$.
   Taking $(x_0, y_0, z_0) = (0, -2, 5)$, the equation we are looking for is $6(x-0) - 22(y+2) - 29(z-5) = 0$ or $6x - 22y - 29z = -101$.

9. Find the point at which the line $x = t - 1$, $y = 1 + 2t$, $z = 3 - t$ intersects the plane $3x - y + 2z = 5$.

   Substitute the parametric equations of the line into the equation of the plane:
   $3(t-1) - (1+2t) + 2(3-t) = 5 \Rightarrow -t + 2 = 5 \Rightarrow t = -3$. Therefore, the point of

intersection of the line and the plane is given by $x = -3-1 = -4$, $y = 1+2(-3) = -5$, and $z = 3-(-3) = 6$, that is, the point $(-4, -5, 6)$.

B.5 Cylinders and Quadric Surfaces

Definition. Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

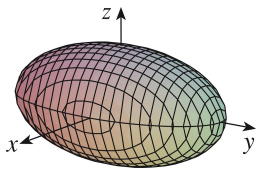
where $A, \ldots, J$ are constants.

Fact. There's too many to list them all. We will look at some examples.

Note. In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curve sare called **traces** (or cross-sections) of the surface.

Fix one of the variables
$x=K$, $y=K$ } vertical traces
$z=K$ horizontal traces
K is constant

Ellipsoid



- The equation is $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$

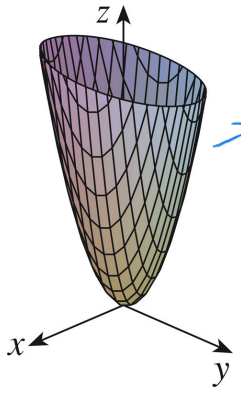
Recall that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of an ellipse

- All traces are ellipses.

If $x=K$, $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{K^2}{a^2}$, which can be rearranged into the equation of an ellipse. Similarly, we obtain ellipses if $y=K$ or $z=K$.

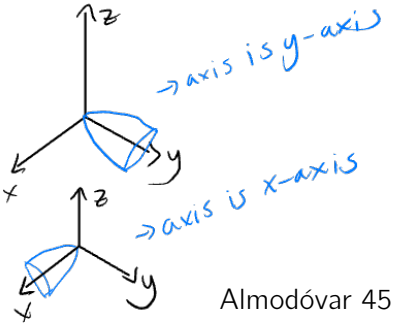
- If $a = b = c$, the ellipsoid is a sphere. $x^2+y^2+z^2=a^2$

Elliptic Paraboloid



→ axis is the z-axis

- The equation is $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ or $\frac{y}{c} = \frac{x^2}{a^2} + \frac{z^2}{b^2}$ or $\frac{x}{c} = \frac{y^2}{a^2} + \frac{z^2}{b^2}$



If we slice the solid horizontally, we obtain an ellipse

- Horizontal traces are ellipses.

If $z=K$, $\frac{K}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2} \Rightarrow 1=\frac{x^2}{\frac{Ka^2}{c}}+\frac{y^2}{\frac{Kb^2}{c}}$ ← equation of an ellipse

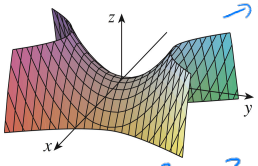
- Vertical traces are parabolas.

If $x=K$ (or $y=K$), $\frac{z}{c}=\frac{K^2}{a^2}+\frac{y^2}{b^2}$ is the equation of a parabola

- The variable raised to the first power indicates the axis of the paraboloid.

Hyperbolic Paraboloid

variable raised to the first power gives the axis



→ axis is z-axis

- The equation is $\boxed{\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

Recall $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are equations of hyperbolas

- Horizontal traces are hyperbolas.

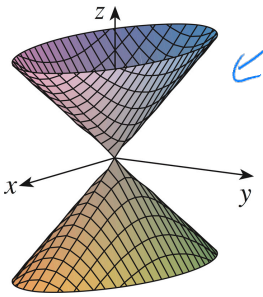
If $z=K$, $\frac{K}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2} \Rightarrow 1=\frac{x^2}{\frac{Ka^2}{c}}-\frac{y^2}{\frac{Ka^2}{c}}$ is the eq. of a hyperbola

- Vertical traces are parabolas.

If $x=K$ (or $y=K$), $\frac{z}{c}=\frac{K^2}{a^2}-\frac{y^2}{b^2}$ is the eq. of a parabola

- The case where $c<0$ is illustrated.

Cone



axis is z-axis

Isolated variable gives the axis of the surface

- The equation is $\boxed{\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}}$
- Horizontal traces are ellipses.

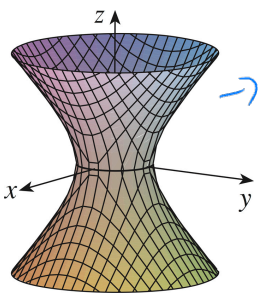
If $z=k$, $\frac{k^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow 1 = \frac{x^2}{\left(\frac{ka}{c}\right)^2} + \frac{y^2}{\left(\frac{kb}{c}\right)^2}$ is an eq. of an ellipse

- Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

If $x=k$, $\frac{z^2}{c^2} = \frac{k^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{z^2}{c^2} - \frac{y^2}{b^2} = \frac{k^2}{a^2}$ ✓ eq. of hyperbola

If $x=0$, $\frac{z^2}{c^2} = \frac{y^2}{b^2}$ we obtain a pair of lines by taking square roots $z = \pm \frac{cy}{b}$

Hyperboloid of One Sheet



→ axis is z-axis

Gives the axis

- The equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} \ominus \frac{z^2}{c^2} = 1$
- Horizontal traces are ellipses.

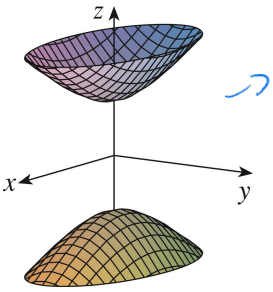
If $z=k$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$, which can be rearranged into an eq. of an ellipse

- Vertical traces are hyperbolas.

If $x=k$, $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}$, which can be rearranged into an eq of a hyperbola

- The axis of symmetry corresponds to the variable whose coefficient is negative.

Hyperboloid of Two Sheets



→ axis is z-axis

Gives the axis

- The equation is $\boxed{-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$
- Horizontal traces in $z = k$ are ellipses.

$z=k \Rightarrow \frac{k^2}{c^2} - 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ ↙ eq of ellipse

- Vertical traces are hyperbolas.

$x=k$ (or $y=k$) $-\frac{k^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{a^2}$

↓ eq of hyperbola

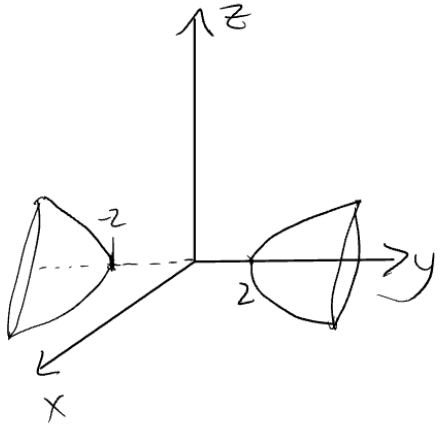
- The two minus signs indicate two sheets.

Example. Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

$4x^2 - y^2 + 2z^2 + 4 = 0$

Divide by 4: $x^2 - \frac{y^2}{4} + \frac{z^2}{2} + 1 = 0$

$1 = -x^2 + \frac{y^2}{4} - \frac{z^2}{2}$



Hyperboloid of two sheets
axis is y-axis
vertex: (0,0,0)

Example. Reduce the equation $y^2 = x^2 + \frac{1}{9}z^2$ to one of the standard forms, classify the surface, and sketch it.

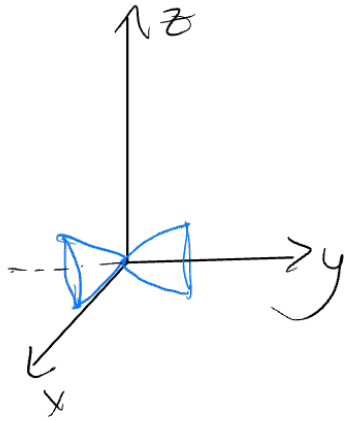
it already is!

Compare to $\frac{y^2}{c^2} = \frac{x^2}{a^2} + \frac{z^2}{b^2}$

Cone! with $a=c=1$ and $b=3$

axis of direction is y-axis

vertex is $(0,0,0)$



Example. Reduce the equation $4x^2 - y + 2z^2 = 0$ to one of the standard forms, classify the surface, and sketch it.

$4x^2 - y + 2z^2 = 0$

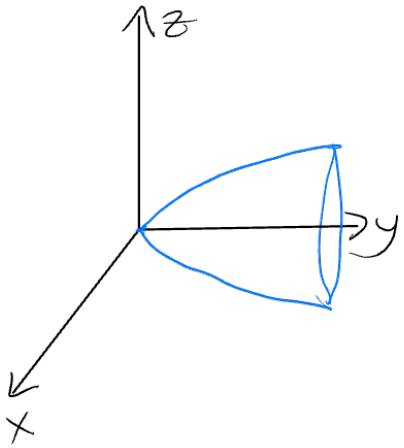
$x^2 - \frac{y}{4} + \frac{z^2}{2} = 0$

$x^2 + \frac{z^2}{2} = \frac{y}{4}$

elliptic paraboloid

axis is y-axis

vertex is $(0,0,0)$



$x^2 - y^2 - z^2 - 4x - 2z + 3 = 0$

Example. Reduce the equation $y^2 = x^2 + \frac{1}{9}z^2$ to one of the standard forms, classify the surface, and sketch it.

Complete the square! $x^2 - 4x - y^2 - z^2 - 2z + 3 = 0$

$x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 - y^2 - (z^2 + 2z + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2) + 3 = 0$

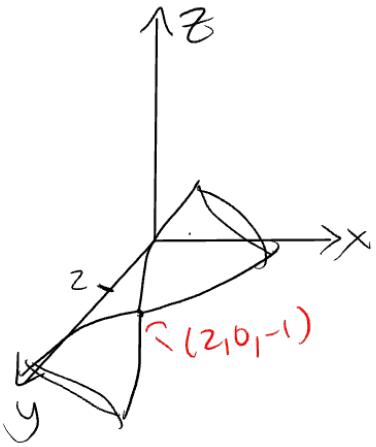
$(x-2)^2 - 4 - y^2 - (z+1)^2 + 1 + 3 = 0$

$(x-2)^2 - y^2 - (z+1)^2 = 0$

$(x-2)^2 = y^2 + (z+1)^2$ cone!

axis of direction is parallel to the x-axis and it's the line $y=0$, $z=-1$

vertex: $(2, 0, -1)$



Example. Reduce the equation $4x^2 - y + 2z^2 = 0$ to one of the standard forms, classify the surface, and sketch it.

$x^2 - y^2 + z^2 - 4x - 2z = 0$

$x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 - y^2 + z^2 - 2z + \left(\frac{-2}{2}\right)^2 - \left(\frac{-2}{2}\right)^2 = 0$

$(x-2)^2 - 4 - y^2 + (z-1)^2 - 1 = 0$

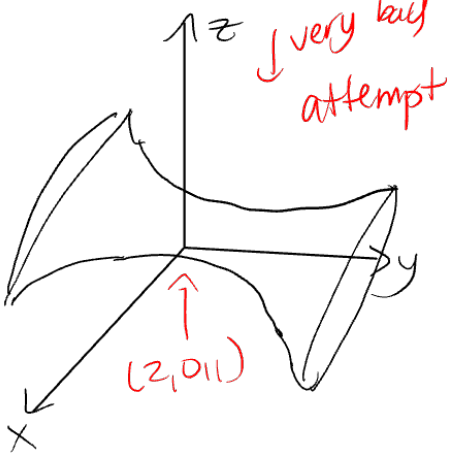
$(x-2)^2 - y^2 + (z-1)^2 = 5$

$\frac{(x-2)^2}{5} - \frac{y^2}{5} + \frac{(z-1)^2}{5} = 1$

Hyperboloid of one sheet

Axis of direction is parallel to y-axis and is the line $x=2$, $z=1$

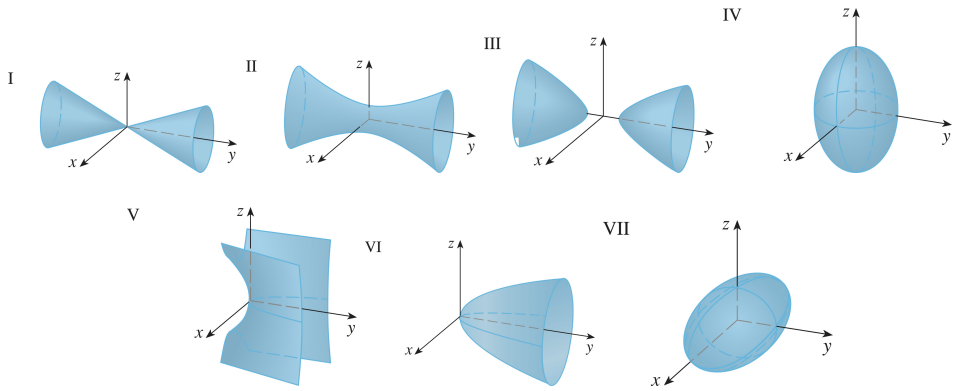
vertex is $(2, 0, 1)$



Homework B.5: Quadric Surfaces

Prof. Leyda Almodóvar

Name: _____



1. Match the equation with its graph (labeled I-VII). Give reasons for your choice.

   (a) $x^2 + 4y^2 + 9z^2 = 1$

   This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \frac{y^2}{(\frac{1}{2})^2} + \frac{z^2}{(\frac{1}{3})^2} = 1$, with x-intercepts ± 1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

   (b) $9x^2 + 4y^2 + z^2 = 1$

   This is the equation of an ellipsoid: $9x^2 + 4y^2 + z^2 = \frac{x^2}{(\frac{1}{3})^2} + \frac{y^2}{(\frac{1}{2})^2} + z^2 = 1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ± 1. So the major axis is the x-axis and the only possible graph is IV.

   (c) $x^2 - y^2 + z^2 = 1$

   This is the equation of a hyperboloid of one sheet, with $a = b$, $c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis, hence the correct graph is II.

(d) $-x^2 + y^2 - z^2 = 1$

This is a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis and the graph is III.

(e) $y = 2x^2 + z^2$

There are no real values of x and z that satisfy this equation for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

(f) $y^2 = x^2 + 2z^2$

This is the equation of a cone with axis the y-axis, so the graph is I.

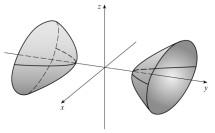
(g) $y = x^2 - z^2$

This is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

2. Reduce the equation to one of the standard forms, classify the surface, and sketch it.

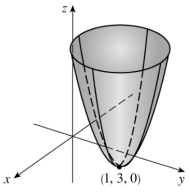
(a) $y^2 = x^2 + 4z^2 + 4$

$y^2 = x^2 + 4z^2 + 4$ is equivalent to $-x^2 + y^2 - 4z^2 = 4$ or $-\frac{x^2}{4} + \frac{y^2}{4} - z^2 = 1$ represents a hyperboloid of two sheets with axis the y-axis.



(b) $x^2 + y^2 - 2x - 6y - z + 10 = 0$

Completing the squares in x and y gives $(x^2 - 2x + 1) + (y^2 - 6y + 9) - z = 0$, which implies $(x - 1)^2 + (y - 3)^2 - z = 0$ or $z = (x - 1)^2 + (y - 3)^2$, a circular paraboloid with vertex $(1, 3, 0)$ and axis the vertical line $x = 1$, $y = 3$.



B.6 Functions of Multiple Variables

In the previous section we studied one particular type of function of multiple variables – planes, which are *linear* functions in two variables. This section is devoted to expanding our horizons to other types of multivariable functions.

Definition. A **function f of two variables** is a rule that assigns each coordinate pair (x, y) to a unique value $z = f(x, y)$. The domain of f is the set of all input pairs (x, y), and the range of f is the set of all assigned z values.

Definition. The **graph** of a function of two variables, f, is the set of all points (x, y, z) such that $z = f(x, y)$. This is usually a surface in 3-space.

Example. Let $f(x, y) = x^2 + y^2$. Consider the following:

Compare to $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

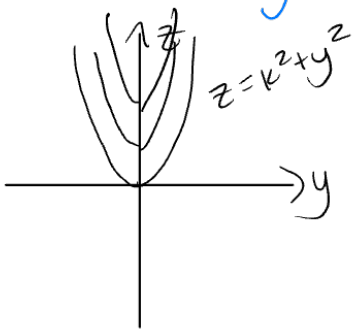
1. Does this function ever have a negative output? (Negative z-values.)
2. What is the domain? The range?
3. What does this look like near $(0, 0)$?
4. What if you fix an x? Fix a y? Sketch those graphs in 2-space.
5. What will this look like far away from $(0, 0)$?

1) No, because $x^2 \geq 0$ and $y^2 \geq 0$.

2) Domain is all pairs (x, y)
Range is $z \geq 0$

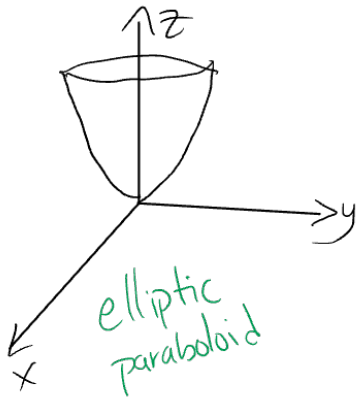
3) If x and y are small, close to 0, then so is z.

4) If $x = k$, k a constant
$z = k^2 + y^2$ → parabolas



If $y = k$, k constant
$z = x^2 + k^2$
parabolas

5) Far away from $(0,0)$ z gets large
If $z = k$, k constant
$k = x^2 + y^2$ ← circle of radius $\sqrt{k}$ centered at $(0,0)$



Example. Now try to graph $f(x,y) = \dfrac{1}{x^2+y^2}$, using the same set of questions.

1) No because $x^2 \geq 0$ and $y^2 \geq 0$.

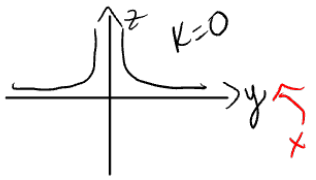
2) Domain: all $(x,y) \neq (0,0)$

Range: $z > 0$

3) Near $(0,0)$ z gets very large

4) Fix $x = k$, k is a constant

$z = \dfrac{1}{k^2+y^2}$



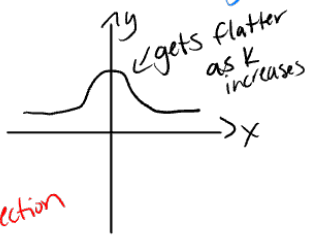
x cross section

Similarly, if $y = k$, we obtain $z = \dfrac{1}{x^2+k^2}$

5) Far from $(0,0)$, z gets small

$z = k$, $k = \dfrac{1}{x^2+y^2} \Rightarrow x^2+y^2 = \dfrac{1}{k}$

circles of radius $\dfrac{1}{\sqrt{k}}$



Definition. Cross-sections are slices of the graph of a function of two variables obtained by fixing x or y, $x = k$ or $y = k$ k constant

Note. The sketches on the yz axes are **x cross sections** and sketches on the xz axes are **y cross sections**.

Example. Describe/sketch the graph of $f(x,y) = xy$. Draw a few x and y cross sections to help you.

Note domain is all (x,y)

Range is all z

$x = k$

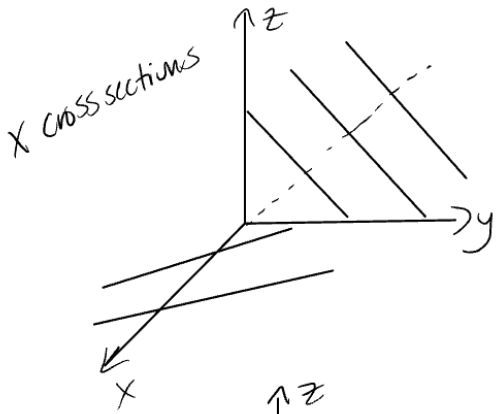
$z = ky$

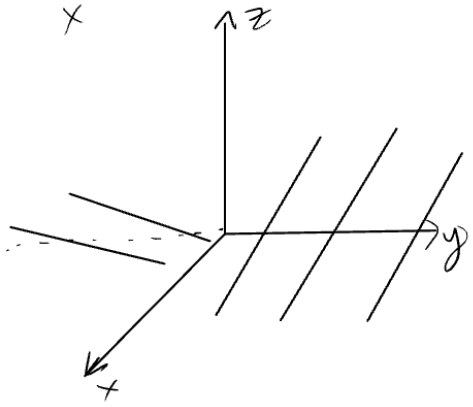
If $k = 0$, $z = 0$ is xy-plane

If $k > 0$, $z = ky$ is a linear function with positive slope

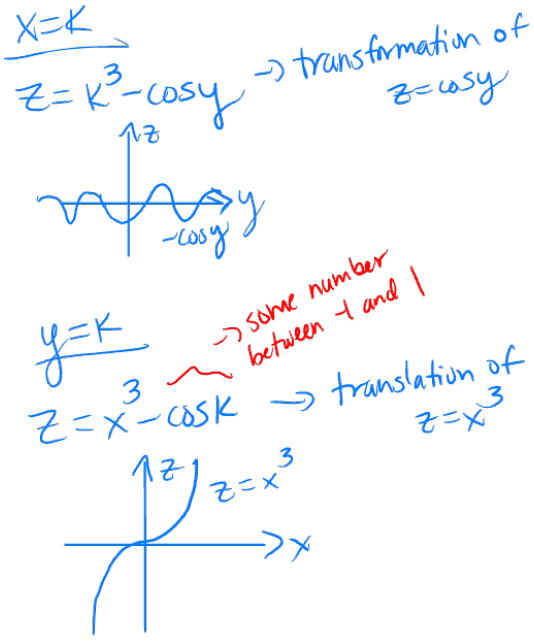
If $k < 0$, $z = ky$ is a linear function with negative slope

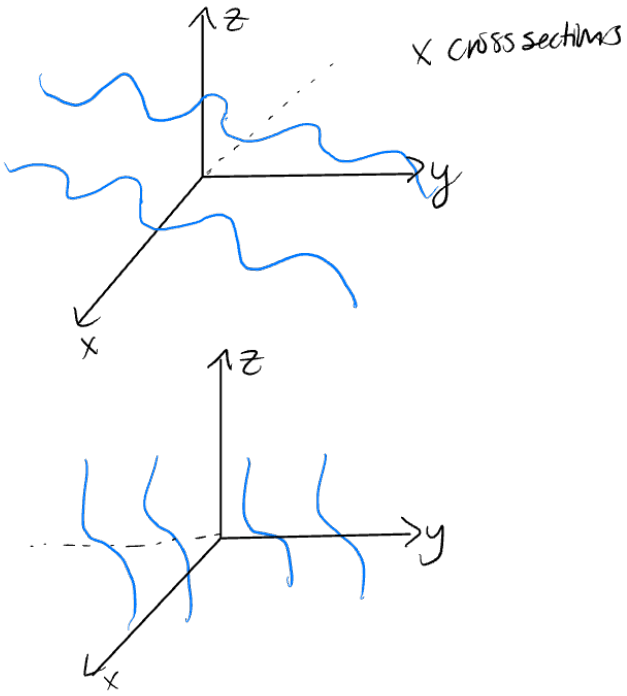
If $y = k$, we obtain similar crosssections





Example. Sketch/describe the graph of $f(x, y) = x^3 - \cos(y)$. Sketch some cross-sections.





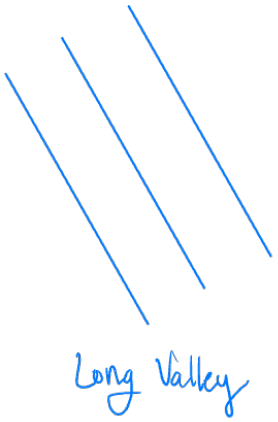
Contours

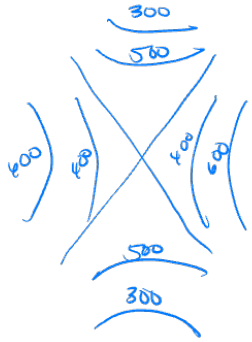
Definition. The **contours** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Contour lines/curves, or "level curves", show "elevation," or what the graph looks like at different z values. These are a complement to cross sections. They also provide information about a 3D surface using only a 2D picture.

Note. "Contour lines" = "level curves" = "level sets."

Example. A few basic contour diagrams:





Note. Algebraically, you can set a z value of k ($k = f(x, y)$) to solve for specific contour equations.

Example. Find and graph several contours of $f(x,y) = x^2 + 2y^2$.

Note $z \geq 0$

$z=1$: $1 = x^2 + 2y^2 = x^2 + \frac{y^2}{1/2}$

ellipse centered at $(0,0)$

$a=1$, $b=1/\sqrt{2}$

$z=2$: $2 = x^2 + 2y^2 \Rightarrow 1 = \frac{x^2}{2} + y^2$

ellipse centered at $(0,0)$

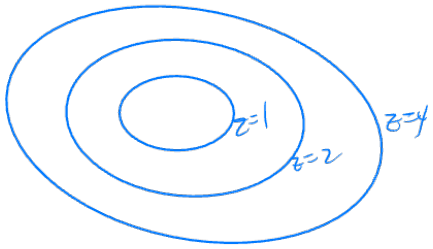
$a=\sqrt{2}$, $b=1$

$z=4$: $4 = x^2 + 2y^2$

$1 = \frac{x^2}{4} + \frac{y^2}{2}$

ellipse centered at $(0,0)$

$a=2$, $b=\sqrt{2}$



Example. Find and graph several contours of $f(x,y) = y - x^2$.

$z=1$: $1 = y - x^2$

$1 + x^2 = y$

$z=0$: $0 = y - x^2$

$x^2 = y$

$z=-1$: $-1 = y - x^2$

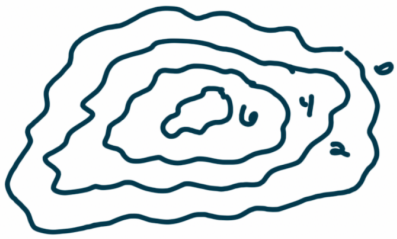
$y = x^2 - 1$

Optional activity to do on your own

Lab 4: Elevate

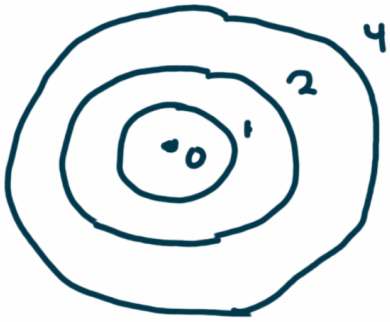
Below are sketches of several contour diagrams / topographical maps. For each of the items below, using play-doh, mold a surface that could by represented by the contours shown.

1.

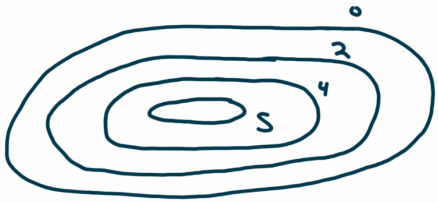


2.

3.

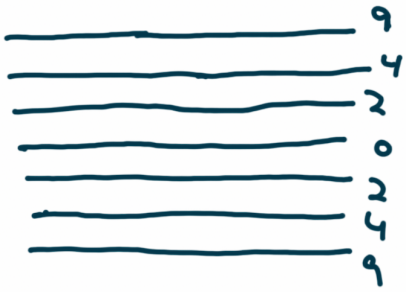


4.



5.

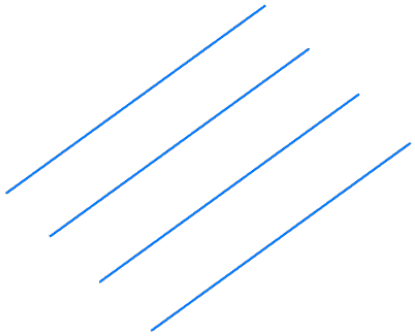
6.



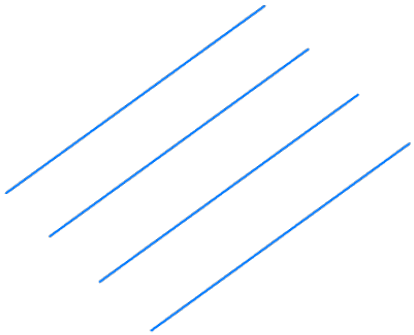
New Facts about Linear Functions / Planes:

- A linear function's cross sections are parallel straight lines with even spacing.
- A linear function's contour diagram also consists of parallel lines with even spacing.
- Even spacing is equivalent to the slope being constant in both the x and y directions, which must be true if the function is linear.

Cross sections for linear functions:



Contour diagrams for linear functions:

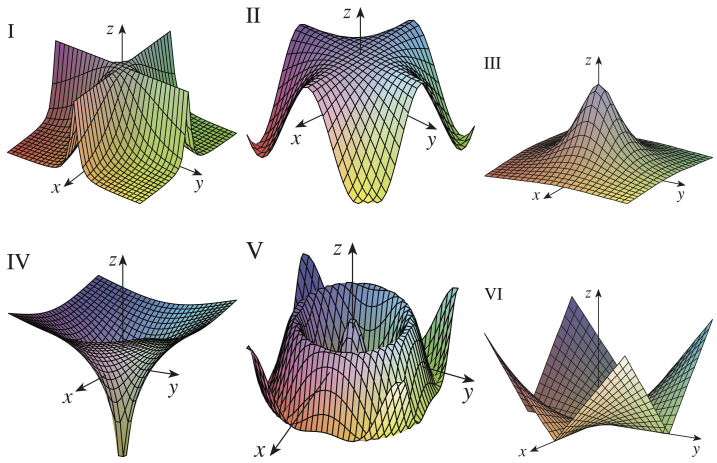


Homework B.6: Functions of Multiple Variables

Prof. Leyda Almodóvar

Name: _____

1. Match the function with its graph (labeled I-VI). Give reasons for your choices.



(a) $f(x,y) = \frac{1}{1+x^2+y^2}$

The trace in $x = 0$ is $z = \frac{1}{1+y^2}$, and the trace in $y = 0$ is $z = \frac{1}{1+x^2}$. The only possibility is graph III. Notice also that the level curves of f are $k = \frac{1}{1+x^2+y^2}$, which implies $x^2 + y^2 = \frac{1}{k} - 1$, a family of circles.

(b) $f(x,y) = \frac{1}{1+x^2y^2}$

The trace in $x = 0$ is the horizontal line $z = 1$, and the trace in $y = 0$ is also $z = 1$. Both graphs I and II have these traces; however, notice that here $z > 0$, so the graph is I.

(c) $f(x,y) = \ln(x^2 + y^2)$

The trace in $x = 0$ is $z = \ln y^2$, and the trace in $y = 0$ is $z = \ln x^2$. The level curves of f are $k = \ln(x^2 + y^2)$, which implies $x^2 + y^2 = e^k$, a family of circles. In addition, f is large negative when $x^2 + y^2$ is small, so this is graph IV.

(d) $f(x,y) = \cos\sqrt{x^2+y^2}$

The trace in $x = 0$ is $z = \cos\sqrt{y^2} = \cos|y| = \cos y$ by properties of cosine, and the trace in $y = 0$ is $z = \cos\sqrt{x^2} = \cos|x| = \cos x$. Notice also that the level curve $f(x,y) = 0$ is $\cos\sqrt{x^2+y^2} = 0$, which implies $x^2 + y^2 = \left(\frac{\pi}{2} + n\pi\right)^2$, a family of circles, so this is graph V.

(e) $f(x,y) = |xy|$

The trace in $x = 0$ is $z = 0$, and the trace in $y = 0$ is $z = 0$, so it must be graph VI.

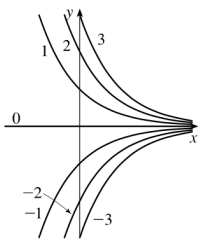
(f) $f(x,y) = \cos(xy)$

The trace in $x = 0$ is $z = \cos 0 = 1$, and the trace in $y = 0$ is $z = 1$. As mentioned in part (b), these traces match both graphs I and II. Here z can be negative, so the graph is II. We could also notice that the trace in $x = 1$ is $z = \cos y$, and the trace in $y = 1$ is $z = \cos x$.

2. Draw a contour map of the function showing several fixed curves.

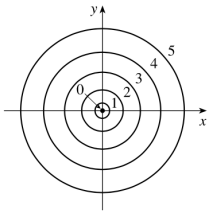
(a) $f(x,y) = ye^x$

The level curves are $ye^x = k$ or $y = ke^{-x}$, a family of exponential curves.



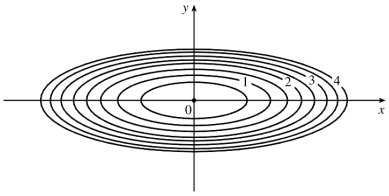
(b) $f(x,y) = \sqrt[3]{x^2+y^2}$

The level curves are $k = \sqrt[3]{x^2+y^2}$ or $x^2 + y^2 = k^3$ $(k \geq 0)$, a family of circles centered at the origin with radius $k^{\frac{3}{2}}$.

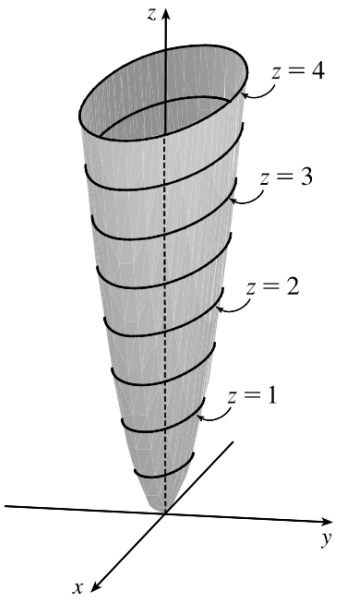


3. Sketch both a contour map and a graph of the function $f(x, y) = x^2 + 9y^2$ and compare them.

The contour map consists of level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.)



If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f. The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.

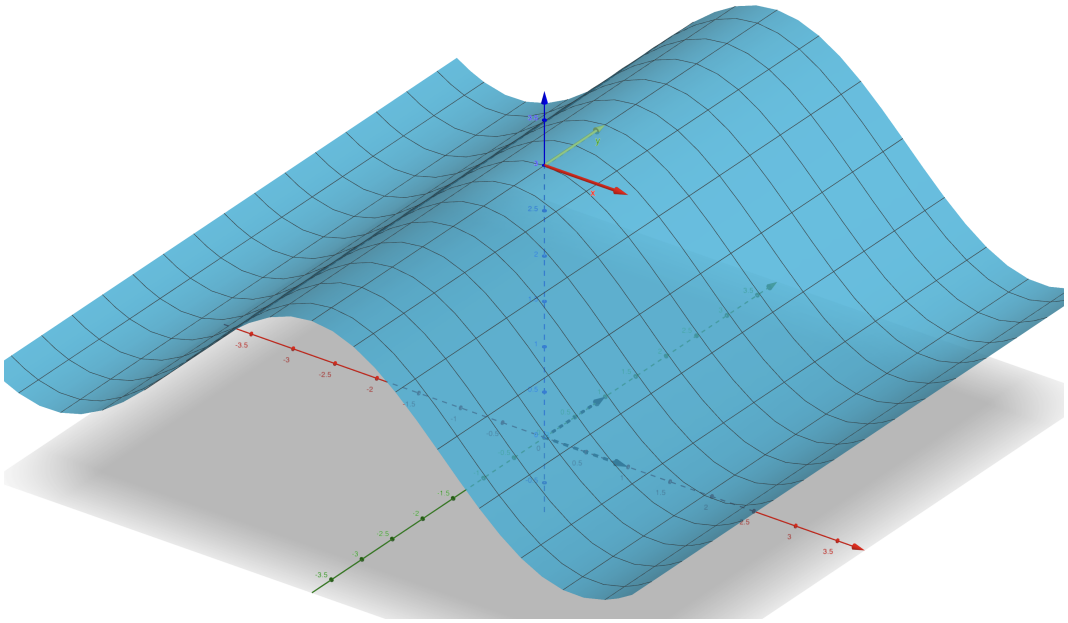


C.1 Partial Derivatives

Recall that the derivative is not just a computation, but has an important meaning for functions.

Definition. For a function of one variable, $f(x)$, the **derivative** $f'(x)$ is the function giving the instantaneous rate of change of f at value in the domain, x. That is, $f'(a)$ is the instantaneous rate of change (i.e. the slope) of f at $x = a$.

When discussing functions of one variable, there is only one direction in which to "move" to consider the slope of the function – the x direction. Since the derivative is the slope, it is the change in the function's value over the change in the input variable's value: $\frac{\Delta f}{\Delta x}$ or $\frac{\Delta y}{\Delta x}$. When we consider functions of more than one variable, we then have more than one direction in which to "move" and calculate the slope – the obvious two being the x direction and the y direction.



Definition. For all points at which the limits exist, we define the **partial derivatives with respect to x and y** at the point (a, b) by

$$f_x(a,b) = \text{ Rate of change of } f \text{ with respect to } x \text{ at the point } (a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \text{ Rate of change of } f \text{ with respect to } y \text{ at the point } (a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

If we let a and b vary, we have the **partial derivative functions:**

$$f_x(x,y) = \frac{\partial z}{\partial x} \text{ and } f_y(x,y) = \frac{\partial z}{\partial y}$$

Note the difference in notation: $\frac{dy}{dx}$ ← derivative

partial derivative of f with respect to x. Also denoted f_x

partial derivative of f with respect to y. Also denoted f_y

Computing Partial Derivatives

We use all the same rules as derivatives from single-variable calculus, with these as additions:

- When computing a derivative with respect to x, treat y as a constant.
- When computing a derivative with respect to y, treat x as a constant.

Example. $z = (3xy + 2x)^5$. Compute both partial derivatives.

$$\frac{\partial z}{\partial x} = 5(3xy+2x)^4 \cdot \frac{\partial}{\partial x}(3xy+2x) = \boxed{5(3xy+2x)^4 \cdot (3y+2)}$$

$$\frac{\partial z}{\partial y} = 5(3xy+2x)^4 \cdot \frac{\partial}{\partial y}(3xy+2x) = 5(3xy+2x)^4 \cdot (3x+0)$$
$$= \boxed{15x(3xy+2x)^4}$$

Fact. Geometric interpretation of partial derivatives:

- f_x is the slope in the x direction (change in z with respect to change in x).
- f_y is the slope in the y direction (change in z with respect to change in y).

Second Order Partial Derivatives

Second order partial derivatives are just the result of taking a derivative a second time. For $z = f(x, y)$, here are all four possible second order partials:

First find f_x, then find $\frac{\partial}{\partial y}(f_x)$

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = (f_x)_x, \qquad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y, \qquad \frac{\partial^2 z}{\partial y^2} = f_{yy} = (f_y)_y$$

Note: f_{xy} and f_{yx} are called mixed partial derivatives

First find f_y, then find $\frac{\partial}{\partial x}(f_y)$

Example. Compute f_{xy} for $f(x, y) = (x - y + 1)^2$.

First we find f_x

$$f_x = 2(x-y+1) \cdot \frac{\partial}{\partial x}(x-y+1) = 2(x-y+1) \cdot 1 = 2(x-y+1)$$

$$f_{xy} = \frac{\partial}{\partial y}\left[2(x-y+1)\right] = \boxed{-2}$$

Example. Compute f_{xx} for $f(x,y) = e^{-2x^2-y^2}$.

$f_x = e^{-2x^2-y^2} \cdot \frac{\partial}{\partial x}(-2x^2-y^2) = -4x \cdot e^{-2x^2-y^2}$

$f_{xx} = -4\left[(x)' e^{-2x^2-y^2} + (e^{-2x^2-y^2})' x\right] = \boxed{-4\left[e^{-2x^2-y^2} - 4x^2 e^{-2x^2-y^2}\right]}$

Example. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 + 2y^2 + 3z^2 = 1$

$\frac{\partial}{\partial x}(x^2+2y^2+3z^2) = \frac{\partial}{\partial x}(1)$

$2x + 0 + 6z\frac{\partial z}{\partial x} = 0$

$6z\frac{\partial z}{\partial x} = -2x$

$\boxed{\frac{\partial z}{\partial x} = \frac{-x}{3z}}$

$\frac{\partial}{\partial y}(x^2+2y^2+3z^2) = \frac{\partial}{\partial y}(1)$

$0 + 4y + 6z \cdot \frac{\partial z}{\partial y} = 0$

$6z\frac{\partial z}{\partial y} = -4y$

$\boxed{\frac{\partial z}{\partial y} = \frac{-2y}{3z}}$

Worksheet: Partial Derivative Practice

Theorem (**Equality of Mixed Partial Derivatives - Clairaut's Theorem**).
If f_{xy} and f_{yx} are continuous at all (a,b) in their domain, then

$$\boxed{f_{yx} = f_{xy}}$$

Note. This theorem means you only need to compute one way or the other, not in both orders.

Earlier we considered $f(x,y) = (x-y+1)^2$ and found $f_{xy} = -2$

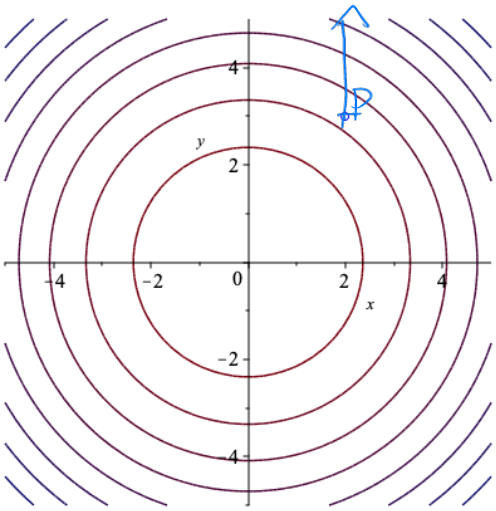
$f_y = 2(x-y+1) \cdot \frac{\partial}{\partial y}(x-y+1) = -2(x-y+1) \Rightarrow f_{yx} = -2(1-0+0) = -2$

Fact. Geometric interpretation of second-order derivatives:

- f_{xx} is the change of the *slope* in the x direction with respect to x (concavity in the x direction).
- f_{yy} is the change of the *slope* in the y direction with respect to y (concavity in the y direction).
- f_{xy} is the change of the *slope* in the x direction as one moves in the y direction.

red = high value
blue = low value

Example. For each of the given contour diagrams, determine whether f_x, f_y, f_{xx}, f_{yy}, and f_{xy} are positive, negative or zero at the given point.



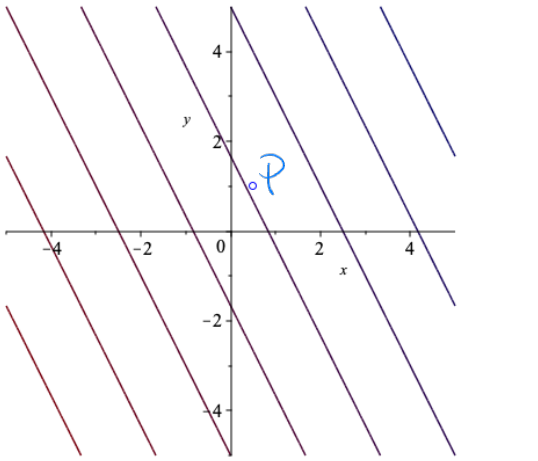
• f_x: fix y, let x vary. Note that the values of f go from high to low as we move from the left of P to the right.
$f_x < 0$

• f_y: fix x, let y vary. The values of f go from high to low as we move in the positive y-direction $\Rightarrow f_y < 0$

• f_{xx}: the rate of change of f_x as we move in the positive x-direction.
The level curves are closer to the right of P than to the left of P
$\Rightarrow$ f is decreasing more quickly to the right of P
$\Rightarrow f_{xx} < 0$

• $f_{yy} < 0$, similar reasoning as f_{xx}

• f_{xy}: How is the steepness changing in the x-direction as we increase y?
f_x is decreasing more quickly as we increase y $\Rightarrow f_{xy} < 0$



red purple blue

• $f_x < 0$

• $f_y < 0$

• $f_{xx} = 0$. Note the level curves are equally spaced as we go from left to right.

• $f_{yy} = 0$. Level curves are equally spaced as we move in the y-direction.

• $f_{xy} = 0$

Homework C.1 - Solutions

Prof. Leyda Almodóvar

Name: _____

1. Find the first partial derivatives of the function.

   (a) $f(x,y) = x^2y - 3y^4$

   $f_x = 2x \cdot y - 0 = \boxed{2xy}$

   $f_y = x^2 \cdot 1 - 3 \cdot 4y^3 = \boxed{x^2 - 12y^3}$

   (b) $f(x,t) = \sqrt{3x+4t}$

   $f_x = \frac{1}{2}(3x+4t)^{-\frac{1}{2}}(3) = \boxed{\frac{3}{2\sqrt{3x+4t}}}$

   $f_t = \frac{1}{2}(3x+4t)^{-\frac{1}{2}}(4) = \boxed{\frac{2}{\sqrt{3x+4t}}}$

   (c) $z = x\sin(xy)$

   $\frac{\partial z}{\partial x} = x \cdot [\cos(xy)](y) + [\sin(xy)] \cdot 1 = \boxed{xy\cos(xy) + \sin(xy)}$

   $\frac{\partial z}{\partial y} = x[\cos(xy)](x) = \boxed{x^2\cos(xy)}$

   (d) $f(x,y) = \frac{x}{(x+y)^2}$

   $f_x = \frac{(x+y)^2(1)-(x)(2)(x+y)}{[(x+y)^2]^2} = \frac{(x+y)[x+y-2x]}{(x+y)^4} = \boxed{\frac{y-x}{(x+y)^3}}$

   $f_y = \frac{(x+y)^2(0)-(x)(2)(x+y)}{[(x+y)^2]^2} = \boxed{-\frac{2x}{(x+y)^3}}$

   (e) $u(r,\theta) = \sin(r\cos\theta)$

   $u_r = \cos(r\cos\theta) \cdot \cos\theta = \boxed{\cos\theta\cos(r\cos\theta)}$

   $u_\theta = \cos(r\cos\theta)(-r\sin\theta) = \boxed{-r\sin\theta\cos(r\cos\theta)}$

(f) $f(x,y) = x^y$

$f_x = \boxed{yx^{y-1}}$

$f_y = \boxed{x^y \ln x}$

(g) $u = x^{\frac{y}{z}}$

$u_x = \boxed{\frac{y}{z}x^{(\frac{y}{z})-1}}$

$u_y = x^{\frac{y}{z}} \ln x \cdot \frac{1}{z} = \boxed{\frac{x^{\frac{y}{z}} \cdot \ln x}{z}}$

$u_z = x^{\frac{y}{z}} \ln x \cdot \frac{-y}{z^2} = \boxed{-\frac{yx^{\frac{y}{z}}}{z^2} \ln x}$

2. Find $f_y\left(1, \frac{1}{2}\right)$ for $f(x,y) = y\sin^{-1}(xy)$.

$f_y(x,y) = y \cdot \frac{1}{\sqrt{1-(xy)^2}}(x) + \sin^{-1}(xy) \cdot 1 = \frac{xy}{\sqrt{1-x^2y^2}} + \sin^{-1}(xy)$, so

$f_y\left(1, \frac{1}{2}\right) = \frac{1 \cdot \frac{1}{2}}{\sqrt{1-1^2(\frac{1}{2})^2}} + \sin^{-1}(1 \cdot \frac{1}{2}) = \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} + \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{1}{\sqrt{3}} + \frac{\pi}{6}}$

3. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^2 - y^2 + z^2 - 2z = 4$.

$\frac{\partial}{\partial x}(x^2 - y^2 + z^2 - 2z) = \frac{\partial}{\partial x}(4) \quad \Rightarrow \quad 2x - 0 + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad (2x-2)\frac{\partial z}{\partial x} = -2x$

$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-2x}{2z-2} = \boxed{\frac{x}{1-z}}$ and

$\frac{\partial}{\partial y}(x^2 - y^2 + z^2 - 2z) = \frac{\partial}{\partial y}(4) \quad \Rightarrow \quad 0 - 2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad (2z-2)\frac{\partial z}{\partial y} = 2y$

$\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{2y}{2z-2} = \boxed{\frac{y}{z-1}}$

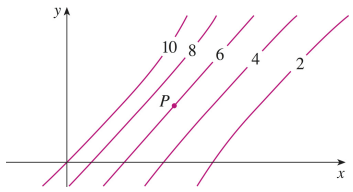
4. Find all the second partial derivatives for $f(x,y) = x^4y - 2x^3y^2$.

$f_x = \boxed{4x^3y - 6x^2y^2} \quad \Rightarrow \quad f_{xx} = \boxed{12x^2y - 12xy^2}$

$f_y = \boxed{x^4 - 4x^3y} \quad \Rightarrow \quad f_{yy} = \boxed{-4x^3}$

$f_{xy} = f_{yx} = \boxed{4x^3 - 12x^2y}$

5. Level curves are shown for a function f. Determine whether the following partial derivatives are positive or negative at the point P.



(a) f_x

If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so $f_x < 0$ at P.

(b) f_y

If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so $f_y > 0$ at P.

(c) f_{xx}

$f_{xx} = \frac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\frac{\partial}{\partial x}(f_x) = f_{xx} > 0$ at P.

(d) f_{xy}

$f_{xy} = \frac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence $f_{xy} < 0$.

(e) f_{yy}

$f_{yy} = \frac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $f_{yy} > 0$ at P.

C.2 The Chain Rule

In this section we will learn how to take derivatives of functions of multiple variables when those variables are also given as functions.

Example. Consider the function:

$$z = f(g(t), h(t))$$

This is a function $f(x, y)$ where $x = g(t), y = h(t)$. That is the variables x and y are now functions of a variable t.

Example. Consider the function:

$$z = f(g(u, v), h(u, v))$$

This is a function $f(x, y)$ where $x = g(u, v), y = h(u, v)$. That is the variables x and y are now functions of two variables u, v.

Note. These are just compositions of functions, as you learn about in Pre-Calculus or Calculus I. We use the chain rule for derivatives of compositions of functions. This is no different from the concept in Calculus I, just with an added level of complexity.

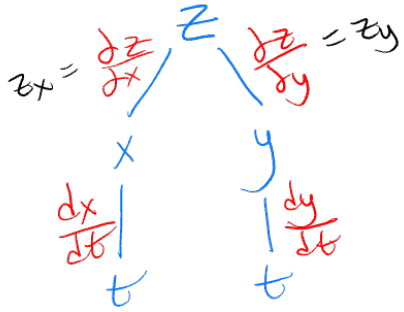
The Chain Rule for $\mathbf{z = f(x, y), x = g(t), y = h(t)}$:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

regular derivatives

partial derivatives

It helps to make a diagram:



Example. Use the Chain Rule to find $\frac{dz}{dt}$ for $z = \sin x \cos y$, $x = \sqrt{t}$, $y = \frac{1}{t}$.

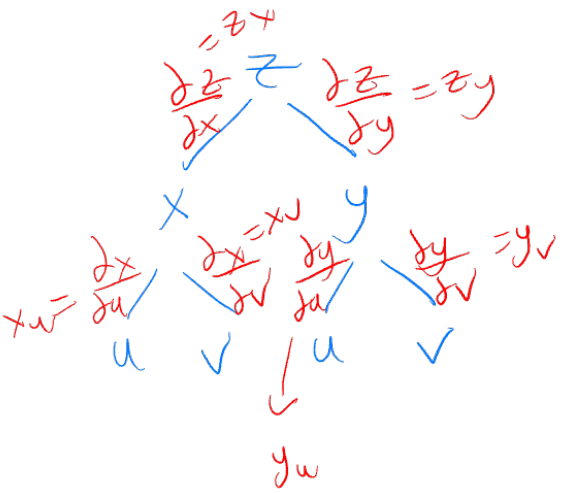
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (\cos x \cos y)\cdot\frac{1}{2}t^{-1/2} + (-\sin x \sin y)\cdot\left(\frac{-1}{t^2}\right)$$

$$= \boxed{(\cos\sqrt{t}\cos\tfrac{1}{t})\cdot\frac{1}{2\sqrt{t}} + \frac{\sin\sqrt{t}\sin\frac{1}{t}}{t^2}}$$

not needed

The answer can be in terms of multiple variables

The Chain Rule for $z = \mathbf{f}(x, y), x = \mathbf{g}(u, v), y = \mathbf{h}(u, v)$:

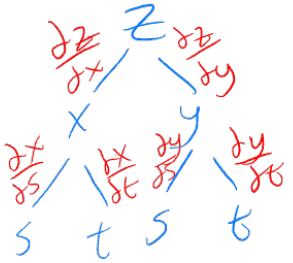


$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

f, x, and y are functions of multiple variables

Notation. We use $\dfrac{\partial z}{\partial x}$ to mean a partial derivative (so z is a function of more than just x). We mean $\dfrac{dx}{dt}$ to mean a total derivative (x is a function of *only* t). We also often use the notation z_u to mean partial derivative of z with respect to u.

Example. Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ for $z = \tan^{-1}(x^2 + y^2)$, $x = s \ln t$, $y = te^s$.



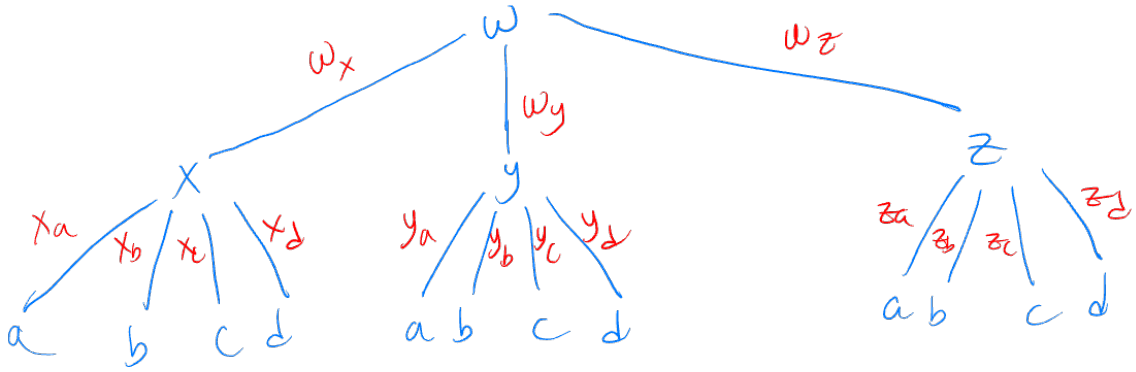
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= \frac{2x}{1+(x^2+y^2)^2} \cdot (\ln t) + \frac{2y}{1+(x^2+y^2)^2} \cdot te^s$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= \frac{2x}{1+(x^2+y^2)^2} \cdot \frac{s}{t} + \frac{2y}{1+(x^2+y^2)^2} \cdot e^s$$

The Diagram Method:

- Draw paths downward for each of the function's variables.
- Multiply along the paths.
- Add together contributions from each path.
- The number of resulting derivatives should be the number of variables in the bottom row of the diagram.

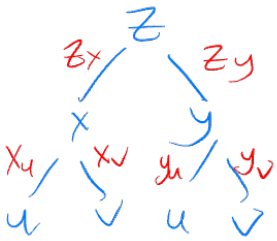
Example. $w = f(x, y, z), x = x(a, b, c, d), y = y(a, b, c, d), z = z(a, b, c, d)$.



Example. $z = f(x, y), x = x(u, v), y = y(u, v), x(2, 3) = 4, y(2, 3) = 6.$

$a = f_x(2, 3), b = f_x(4, 6), c = f_y(2, 3), d = f_y(4, 6),$

$e = x_u(2, 3), f = x_v(2, 3), g = y_u(2, 3), h = y_v(2, 3)$



Find $z_u(2, 3), z_v(2, 3)$.

a and c make no sense!

$z_u = z_x \cdot x_u + z_y \cdot y_u$

$z_v = z_x \cdot x_v + z_y \cdot y_v$

f is a function of x and y

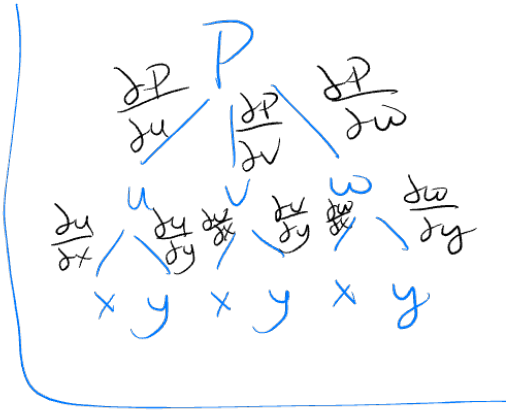
$z_u(2,3) = f_x(4,6) \cdot x_u(2,3) + f_y(4,6) \cdot y_u(2,3)$

$= be + dg$

$z_v(2,3) = f_x(4,6)\, x_v(2,3) + f_y(4,6) \cdot x_v(2,3)$

$= bf + dh$

Example. Given $P = \sqrt{u^2 + v^2 + w^2}$, $u = xe^y$, $v = ye^x$, $w = e^{xy}$, use the Chain Rule to find $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ when $x = 0$ and $y = 2$.



$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial P}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{1}{2}(u^2+v^2+w^2)^{-1/2}\cdot 2u\cdot e^y + \frac{1}{2}(u^2+v^2+w^2)^{-1/2}\cdot 2v\cdot ye^x$$

$$+ \frac{1}{2}(u^2+v^2+w^2)^{-1/2}\cdot 2w\cdot ye^{xy} = \frac{ue^y + vye^x + wye^{xy}}{\sqrt{u^2+v^2+w^2}}$$

If $x=0, y=2 \Rightarrow u=0,\ v=2e^0=2,\ w=e^{0\cdot 2}=1$

$$\left.\frac{\partial P}{\partial x}\right|_{x=0,y=2} = \frac{0+2\cdot(2)e^0+1\cdot 2e^0}{\sqrt{0^2+4+1}} = \boxed{\frac{6}{\sqrt{5}}}$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial P}{\partial z}\cdot\frac{\partial w}{\partial z}$$

$$= \frac{u}{\sqrt{u^2+v^2+w^2}}\left(xe^y\right) + \frac{v}{\sqrt{u^2+v^2+w^2}}\cdot e^x + \frac{w}{\sqrt{u^2+v^2+w^2}}\cdot xe^{xy}$$

$$= \frac{uxe^y + ve^x + wxe^{xy}}{\sqrt{u^2+v^2+w^2}}$$

$$\left.\frac{\partial P}{\partial y}\right|_{x=0,y=2} = \frac{0+2e^0+1\cdot 0e^0}{\sqrt{5}} = \boxed{\frac{2}{\sqrt{5}}}$$

Homework C.2 - The Chain Rule

Prof. Leyda Almodóvar

Name: _____

1. Use the Chain Rule to find $\frac{\mathrm{d}z}{\mathrm{d}t}$ or $\frac{\mathrm{d}w}{\mathrm{d}t}$.

   (a) $z = xy^3 - x^2y$, $x = t^2 + 1$, $y = t^2 - 1$

   $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}t} = (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) = \boxed{2t(y^3 - 2xy + 3xy^2 - x^2)}$

   (b) $z = \frac{x-y}{x+2y}$, $x = e^{\pi t}$, $y = e^{-\pi t}$

   $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(x+2y)(1)-(x-y)(1)}{(x+2y)^2}(\pi e^{\pi t}) + \frac{(x+2y)(1)-(x-y)(2)}{(x+2y)^2}(-\pi e^{-\pi t})$

   $= \frac{3y}{(x+2y)^2}(\pi e^{\pi t}) + \frac{-3x}{(x+2y)^2}(-\pi e^{-\pi t}) = \boxed{\frac{3\pi}{(x+2y)^2}(ye^{\pi t} + xe^{-\pi t})}$

   (c) $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$

   $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}\cdot\cos t + \frac{1}{2}\cdot\frac{2y}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{1}{2}\frac{2y}{x^2+y^2+z^2}\cdot\sec^2 t$

   $= \boxed{\frac{x\cos t - y\sin t + z\sec^2 t}{x^2 + y^2 + z^2}}$

2. Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

   (a) $z = (x - y)^5$, $x = s^2t$, $y = st^2$

   $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 5(x-y)^4(1)\cdot 2st + 5(x-y)^4(-1)\cdot t^2 = \boxed{5(x-y)^4(2st - t^2)}$

   $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 5(x-y)^4(1)\cdot s^2 + 5(x-y)^4(-1)\cdot 2st = \boxed{5(x-y)^4(s^2 - 2st)}$

(b) $z = \sqrt{x}e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot\tfrac{1}{2}x^{-\frac{1}{2}}\right)(t) + \sqrt{x}e^{xy}(x)(2s) = \boxed{\left(yt\sqrt{x} + \frac{t}{2\sqrt{x}} + 2x^{\frac{3}{2}}s\right)e^{xy}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\sqrt{x}\cdot e^{xy}(y) + e^{xy}\cdot\tfrac{1}{2}x^{-\frac{1}{2}}\right)(s) + \sqrt{x}e^{xy}(x)(-2t) = \boxed{\left(ys\sqrt{x} + \frac{s}{2\sqrt{x}} - 2x^{\frac{3}{2}}t\right)e^{xy}}$$

(c) $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-\frac{1}{2}}(2s)$$

$$= te^r\cos\theta - e^r\sin\theta\cdot\tfrac{s}{\sqrt{s^2+t^2}} = \boxed{e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2+t^2}}\sin\theta\right)}$$

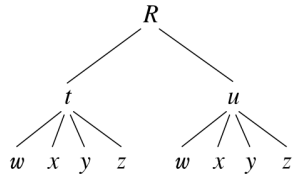
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2+t^2)^{-\frac{1}{2}}(2t)$$

$$= se^r\cos\theta - e^r\sin\theta\cdot\tfrac{t}{\sqrt{s^2+t^2}} = \boxed{e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2+t^2}}\sin\theta\right)}$$

3. Use a tree diagram to write out the Chain Rule for $R = F(t, u)$ where $t = t(w, x, y, z)$, $u = u(w, x, y, z)$.

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial w} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial w} \qquad \frac{\partial R}{\partial x} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial x}$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} \qquad \frac{\partial R}{\partial z} = \frac{\partial R}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial R}{\partial u}\frac{\partial u}{\partial z}$$



4. Use the Chain Rule to find the indicated partial derivatives.

(a) $z = x^4 + x^2y$, $x = s + 2t - u$, $y = stu^2$; $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, $u = 1$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(tu^2)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu)$$

When $s = 4$, $t = 2$, and $u = 1$ we have $x = 7$ and $y = 8$, so

$\frac{\partial z}{\partial s} = (1484)(1) + (49)(2) = \boxed{1582}$

$\frac{\partial z}{\partial t} = (1484)(2) + (49)(4) = \boxed{3164}$

$\frac{\partial z}{\partial u} = (1484)(-1) + (49)(16) = \boxed{-700}$

(b) $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$; $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial \theta}$ when $r = 2$, $\theta = \frac{\pi}{2}$

$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y + z)(\cos\theta) + (x + z)(\sin\theta) + (y + x)(\theta),$

$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} = (y + z)(-r\sin\theta) + (x + z)(r\cos\theta) + (y + z)(r)$

When $r = 2$ and $\theta = \frac{\pi}{2}$ we have $x = 0$, $y = 2$, and $z = \pi$, so

$\frac{\partial w}{\partial r} = (2 + \pi)(0) + (0 + \pi)(1) + (2 + 0)\left(\frac{\pi}{2}\right) = \boxed{2\pi}$ and

$\frac{\partial w}{\partial \theta} = (2 + \pi)(-2) + (0 + \pi)(0) + (2 + 0)(2) = \boxed{-2\pi}$

C.3 Tangent Planes and Linear Approximations

Local linearity is the concept of zooming in far enough to a function that it *appears* linear.

- In single-variable functions (observe Geogebra graph):

$f(x) = \sin x$

- In two-variable functions (observe Geogebra graph):

$f(x,y) = x^3 - \cos y$

If we see something like a plane when we zoom in on a function $f(x,y)$ close to a certain point, this means the function is closely approximated by a linear function $L(x,y)$ at that point. We write: $f(x,y) \approx L(x,y)$. This is very similar to the notion of a *tangent line*. In two-variable calculus we will instead have a **tangent plane.**

- Touches the surface at the point P and gives the direction of the surface at P.
- A tangent plane to a surface S gives an approximation of the function values near P. (at point P)

Lab 5: Tangent Planes

In this lab you will explore equations and graphs of tangent planes.

Answer the following questions by relying on your knowledge of single-variable calculus and the equation of a plane given in a previous section of these notes.

1. In single-variable calculus, what is the slope of the tangent line to the function $f(x)$ given by?

   $m_T = f'(a)$ is the slope of the tangent line to $f(x)$ at $x=a$.

2. What is the equation of a plane? What are the slopes?

   $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where $\vec{n}=\langle a,b,c\rangle$ is a normal vector to the plane and (x_0,y_0,z_0) is a point on the plane. a/c gives the slope in the x-direction and b/c gives the slope in the y-direction.

3. For a function $f(x,y)$, what computation can we perform to determine the slopes in the x and y directions in order to determine the slope of the surface at a point $(a,b,f(a,b))$?

   Partial derivatives $f_x(a,b)$ and $f_y(a,b)$

4. Based on your responses to the previous questions, what do you think the equation of a tangent plane should be?

   $f(x,y)=m(x-x_0)+n(y-y_0)+z_0$

   Replace m with f_x, n with f_y

Definition. Tangent plane to the surface $z=f(x,y)$ at the point (a,b) is given by

$$z = \underline{f_x(a,b)}\,(x-a) + \underline{f_y(a,b)}\,(y-b) + f(a,b)$$

5. Open up a 3D graphing page on *Geogebra* and graph the surface

$$f(x,y) = -\frac{1}{3}x^2 - \frac{1}{4}y^2 + 3$$

6. In a new input box, type the point $P=(1,1,f(1,1))$. (Note: What does $f(1,1)$ give you? Why does it work to type this in this way?)

   The z-value ≈ 2.42

   Because $z=f(x,y)$

7. Find the partial derivatives of $f(x, y)$. Find their values at the point P.

$f_x = -\frac{2}{3}x \qquad f_x(1,1) = -2/3$

$f_y = -\frac{1}{2}y \qquad f_y(1,1) = -1/2$

8. Graph the vector that begins at point P and ends at the point $(2, 1, 1 + f_x(P))$. What does this vector represent?

$Q = (2, 1, 1/3)$

Vector (P, Q)

$\vec{PQ}$ represents the slope of the surface in the x-direction at P.

9. Graph the vector that begins at point P and ends at the point $(1, 2, 1 + f_y(P))$. What does this vector represent?

$R = (1, 2, 1/2)$

Vector (P, R)

$\vec{PR}$ represents the slope of the surface in the y-direction at P.

10. In a new input box, graph the plane with slopes $f_x(P)$ and $f_y(P)$ passing through the point P. This is the **tangent plane** to the surface at the point P. What do you observe? How do the vectors that you graphed relate to the plane? How does the plane relate to the surface?

$z = -\frac{2}{3}(x-1) - 1/2(y-1) + f(1,1)$

The plane contains all the possible tangent lines at P. The tangent plane at P closely approximates the surface near P.

You have just graphed a surface, a tangent plane to the surface at a point, and two vectors representing the slopes of the surface and plane in the x and y directions.

Now, try finding the equation of a couple more tangent planes and graphing the surface, point, and plane together in Geogebra on your own.

For each of the following, find the equation of the tangent plane to the point and graph the function, the point, and the plane.

11. $f(x, y) = \cos(x) - \frac{1}{2}y, P = \left(\frac{\pi}{2}, -2\right)$

12. $f(x, y) = xy + y^3, P = (-3, -1)$

Lab 5 Follow-Up Notes

Definition. Tangent plane to the surface $\mathbf{z = f(x, y)}$ at the point $\mathbf{(a, b)}$ is given by

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Note. In the above equation, $f_x(a, b)$, $f_y(a, b)$, and $f(a, b)$ should all be constants.

Note. The values $f_x(a, b)$ and $f_y(a, b)$ are the slopes in the x- and y-direction of the tangent plane at $(a, b, f(a, b))$. The value of $f(a, b)$ is the z-coordinate.

Since when we zoom in close to a surface $f(x, y)$ it starts to look like its tangent plane, we can approximate $f(x, y)$ values using tangent planes. We call this process **linearization.**

Example. Find the linearization of $f(x, y) = x^3y + 2y$ at the point $(2, 3)$. Use it to estimate $f(2.01, 2.98)$.

$f(2,3) = 2^3 \cdot 3 + 2(3) = 24 + 6 = 30$

Find an equation of the tangent plane at $(2, 3, f(2,3))$

$f_x = 3x^2y \Rightarrow f_x(2,3) = 3(2)^2 \cdot 3 = 36$

$f_y = x^3 + 2 \Rightarrow f_y(2,3) = 2^3 + 2 = 10$

$z = 36(x-2) + 10(y-3) + 30 = L(x,y)$

$f(2.01, 2.98) \approx 36(2.01 - 2) + 10(2.98 - 3) + 30$

$= 36(0.01) + 10(-0.02) + 30$

$= 0.36 - 0.2 + 30$

$= \boxed{30.16}$

Why is linearization useful?

Because it's just easier to work with linear functions.

Also it's useful to solve ODE's.

Homework C.3 - Tangent Planes and Linear Approximations

Prof. Leyda Almodóvar

Name: _____

1. Find an equation of the tangent plane to the given surface at the specified point

   (a) $f(x,y) = (x+2)^2 - 2(y-1)^2 - 5$, $P = (2,3,3)$

   $f_x(x,y) = 2(x+2) \quad \Rightarrow \quad f_x(2,3) = 2(2+2) = 8$

   $f_y(x,y) = -4(y-1) \quad \Rightarrow \quad f_y(2,3) = -4(3-1) = -8$

   An equation of the tangent plane is $z = f_x(2,3)(x-2) + f_y(2,3)(y-3) + 3$

   $\Rightarrow \quad z = 8(x-2) - 8(y-3) + 3 \quad \Rightarrow \quad \boxed{z = 8x - 8y + 11}$

   (b) $f(x,y) = \frac{x}{y^2}$, $P = (-4,2,-1)$

   $f_x(x,y) = \frac{1}{y^2} \quad \Rightarrow \quad f_x(-4,2) = \frac{1}{4}$

   $f_y(x,y) = \frac{-2x}{y^3} \quad \Rightarrow \quad f_y(-4,2) = \frac{8}{8} = 1$

   An equation of the tangent plane is $z = f_x(-4,2)(x+4) + f_y(-4,2)(y-2) - 1$

   $\Rightarrow \quad z = \frac{1}{4}(x+4) + 1(y-2) - 1 \quad \Rightarrow \quad \boxed{z = \frac{1}{4}x + y - 2}$

   (c) $f(x,y) = \ln(x-2y)$, $P = (3,1,0)$

   $f_x(x,y) = \frac{1}{x-2y} \quad \Rightarrow \quad f_x(3,1) = \frac{1}{3-2} = 1$

   $f_y(x,y) = \frac{-2}{x-2y} \quad \Rightarrow \quad f_y(3,1) = \frac{-2}{1} = -2$

   An equation of the tangent plane is $z = f_x(3,1)(x-3) + f_y(3,1)(y-1) + 0$

   $\Rightarrow \quad z = 1(x-3) - 2(y-1) \quad \Rightarrow \quad \boxed{z = x - 2y - 1}$

2. Find the linearization $L(x, y)$ of the function at the given point.

(a) $f(x, y) = \sqrt{xy}$, $(1, 4)$

$f_x(x, y) = \frac{y}{2\sqrt{xy}} \quad \Rightarrow \quad f_x(1, 4) = \frac{4}{2\sqrt{4}} = 1$

$f_y(x, y) = \frac{x}{2\sqrt{xy}} \quad \Rightarrow \quad f_y(1, 4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

The linearization of f at $(1, 4)$ is $L(x, y) = f_x(1, 4)(x-1)+f_y(1, 4)(y-4)+f(1, 4)$

$$\Rightarrow \quad L(x, y) = 1(x - 1) + \tfrac{1}{4}(y - 4) + 2 \quad \Rightarrow \quad \boxed{L(x, y) = x + \frac{1}{4}y}$$

(b) $f(x, y) = \frac{1+y}{1+x}$, $(1, 3)$

$f_x(x, y) = \frac{-(1+y)}{(1+x)^2} \quad \Rightarrow \quad f_x(1, 3) = \frac{-4}{(1+1)^2} = -1$

$f_y(x, y) = \frac{1}{1+x} \quad \Rightarrow \quad f_y(1, 3) = \frac{1}{1+1} = \frac{1}{2}$

The linearization of f at $(1, 3)$ is $L(x, y) = f_x(1, 3)(x-1)+f_y(1, 3)(y-3)+f(1, 3)$

$$\Rightarrow \quad L(x, y) = -1(x - 1) + \tfrac{1}{2}(y - 3) + 2 \quad \Rightarrow \quad \boxed{L(x, y) = -x + \frac{1}{2}y + \frac{2}{3}}$$

C.4 Directional Derivatives and the Gradient Vector

Recall. The *partial derivatives* tell us the rate of change of a function f (that is, the z values) in strictly the x direction or the y direction.

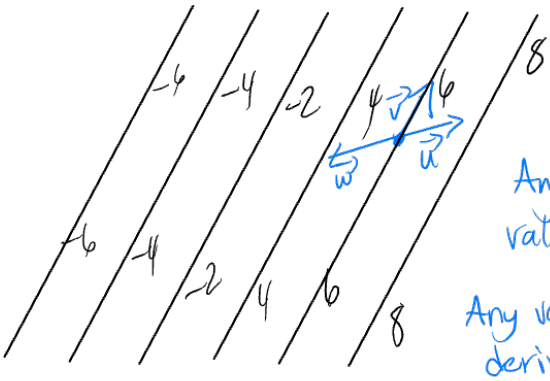
Now we will learn how to compute the rate of change in *any* direction.

Definition. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle u_1, u_2 \rangle$, denoted $D_{\vec{u}} f(x_0, y_0)$, is the rate of change of f at (x_0, y_0) in the direction of $\vec{u}$ and is given by

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided the limit exists. In other words, it is the derivative of f at (x_0, y_0) in the direction of $\vec{u}$.

Example. Directional derivative illustration on a contour diagram:



Any vector that points to a contour line of higher value will give a positive directional derivative.

Any vector that points to a contour line of lower value will give a negative directional derivative.

Any vector parallel to a contour line will give a directional derivative equal to 0.

Is the directional derivative at each point in the direction of the vector shown positive, negative, or zero?

$D_{\vec{u}} f(a,b) > 0$

$D_{\vec{w}} f(a,b) < 0$

$D_{\vec{v}} f(a,b) = 0$

Here is an alternate definition with a much easier formula:

Definition. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle u_1, u_2 \rangle$ is given by

$$\boxed{D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2}$$

Example. 1. Calculate the directional derivative of $f(x, y) = 3x + 7y^2$ at $(1, 1)$ in the direction of $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.

$|\vec{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

$f_x = 3, \quad f_y = 14y$

$f_x(1,1) = 3, \quad f_y(1,1) = 14$

$D_{\vec{u}} f(1,1) = 3 \cdot \frac{3}{5} + 14 \cdot \frac{4}{5}$

$= \frac{9}{5} + \frac{56}{5}$

$= \frac{65}{5} = \boxed{13}$

Slope of f at $(1,1)$ in the direction of $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

$f(x,y)=3x+7y^2$, $(1,1)$

2. Now, calculate in the direction of $\vec{i}+\vec{j}$.

$\vec{i}+\vec{j}=\langle 1,1\rangle \Rightarrow |\langle 1,1\rangle|=\sqrt{2}$

$\vec{v}=\frac{1}{\sqrt{2}}\langle 1,1\rangle=\left\langle \frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle$

$D_{\vec{v}}f=3\cdot\frac{1}{\sqrt{2}}+14\cdot\frac{1}{\sqrt{2}}=\boxed{\frac{17}{\sqrt{2}}}$

↙ slope of f at $(1,1)$ in the direction of $\langle 1,1\rangle$.

Fact. 1. If $\vec{u}=\vec{i}$ then the directional derivative $(D_{\vec{i}}f)$ is $f_x(x_0,y_0)$.

2. If $\vec{u}=\vec{j}$ then the directional derivative $(D_{\vec{j}}f)$ is $f_y(x_0,y_0)$.

If you are not given a *unit* vector, but still asked to calculate the directional derivative, you should:

divide the vector by its magnitude

That is, given $\vec{v}$, $\vec{u}=\frac{\vec{v}}{|\vec{v}|}$

Definition. The **directional derivative** of f in the direction of a unit vector $\vec{u}=\langle u_1,u_2\rangle$, denoted $D_{\vec{u}}f(x,y)$ is given by

$$D_{\vec{u}}f(x,y)=f_x(x,y)u_1+f_y(x,y)u_2$$

Note. This is before a point is plugged in.

Example. Write the general directional derivative for a previous example.

$f(x,y)=3x+7y^2$, $f_x=3$, $f_y=14y$, $\vec{u}=\frac{3}{5}\vec{i}+\frac{4}{5}\vec{j}$

$D_{\vec{u}}f(x,y)=3\cdot\frac{3}{5}+14y\cdot\frac{4}{5}=\boxed{\frac{9}{5}+\frac{56y}{5}}$

Definition. The **gradient vector** of a differentiable function $f(x,y)$ is given by:

$$\text{grad}f=\nabla f=f_x(x,y)\vec{i}+f_y(x,y)\vec{j}$$
$$=\langle f_x(x,y),f_y(x,y)\rangle$$

Notation.

- We also sometimes write $\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}$.
- Note that $\text{grad}f=\nabla f$.

Note: If $f(x,y,z)\Rightarrow \text{grad}f=\nabla f=\langle f_x(x,y,z),f_y(x,y,z),f_z(x,y,z)\rangle$

Example. 1. Find the gradient vector of $f(x,y) = 2x^2 + 3y^2$.

$\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \boxed{\langle 4x, 6y \rangle}$

2. Now, find $\nabla f(2,1)$.

$\nabla f(2,1) = \langle 4(2), 6(1) \rangle = \boxed{\langle 8, 6 \rangle}$

Example. 1. Find the gradient vector of $f(r,h) = 2\pi rh + \pi r^2$.

$\nabla f = \langle f_r(r,h), f_h(r,h) \rangle = \langle 2\pi h + 2\pi r, 2\pi r \rangle$

2. Now, find $\nabla f(2,3)$.

$\nabla f(2,3) = \langle 2\pi(3) + 2\pi(2), 2\pi(2) \rangle = \boxed{\langle 10\pi, 4\pi \rangle}$

Q: What's the relation between ∇f and $D_{\vec{u}} f$?

Fact. The directional derivative of f in the direction of $\vec{u}$ is the dot product of the gradient vector and the unit vector $\vec{u}$.

One more time...

Definition. The **directional derivative** of f in the direction of a unit vector $\vec{u}$, denoted $D_{\vec{u}} f(x,y)$ is given by

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}}$$

Example. Compute the directional derivative of $f(x,y) = 2x^2 + 3y^2$ in the direction of $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$, using the last definition. $|\vec{u}| = 1$

$\nabla f = \langle 4x, 6y \rangle$

$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = 4x \cdot \frac{3}{5} + 6y \cdot \frac{4}{5} = \boxed{\frac{12x}{5} + \frac{24y}{5}}$

Geometric Properties of the Gradient Vector

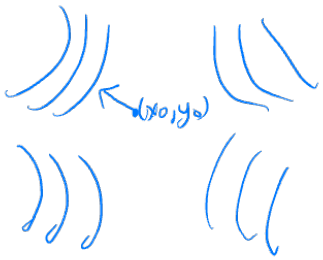
(Hey this is really super important. I'm trying to subtly tell you it's important to learn this. Learn this.)

Suppose we have a function $f(x, y)$ and we consider all possible directional derivatives of f at a given starting point (x_0, y_0). These are all giving rates of change of f in all possible directions away from (x_0, y_0). We might ask: Is there direction that gives the *fastest* rate of change? (a.k.a the biggest directional derivative) What would that rate of change actually be? How do we find it?

$\nabla f(x_0, y_0)$

Fact. The following are geometric properties of ∇f:

- The direction of $\nabla f(x_0, y_0)$ is: → not the surface
  - **Perpendicular** to the contour of f through (x_0, y_0).
  - In the direction of the **maximum rate of change** of f starting from (x_0, y_0).



- The magnitude of $\nabla f(x_0, y_0)$, $|\nabla f(x_0, y_0)|$ is:
  - The **maximum rate of change** of f from that point.
  - Large when contours are close together, small when far apart (indicating the steepness).

$|D_{\vec{u}} f| = |\nabla f \cdot \vec{u}| = |\nabla f||\vec{u}|\cos\theta$, where θ is angle between ∇f and $\vec{u}$

We achieve the highest value when $\cos\theta = 1$
$\Rightarrow \theta = 0$ and ∇f and $\vec{u}$ have the same direction

Lab 6: Directional Derivatives

In this lab you will explore graphs of directional derivatives and gradient vectors.

1. Open up a 3D graphing page on *Geogebra* and graph the surface

$$f(x,y) = -\frac{1}{3}x^2 - \frac{1}{4}y^2 + 3$$

(This should feel familiar from the previous lab.)

2. In a new input box, type the point $P = (1, 1, f(1,1))$.

3. Find $\nabla f(1,1)$.

$\nabla f = \langle -\frac{2}{3}x, -\frac{1}{2}y \rangle$

$\nabla f(1,1) = \langle -2/3, -1/2 \rangle$

4. In a new input box, graph $\nabla f(1,1)$, starting at the point $(1,1,f(1,1))$.

→ Create a new point $Q = (1-2/3, 1-1/2, f(1,1))$ then Vector(P, Q)

5. What does $\nabla f(1,1)$ represent?

The direction of maximum rate of change of f starting at P.

You now have a graph of a surface, a point on the surface, and the gradient vector of the surface at that point. Now, we will use each of these to compute the directional derivative of the surface for a given direction.

6. You will compute and graph the directional derivative of f in the direction of $\vec{v} = \langle 1, 2 \rangle$. Graph $\vec{v}$ in Geogebra, starting at the point $(1, 1, f(1,1))$.

7. Check whether or not $\vec{v}$ is a unit vector, and if it is not, make it one and call it $\vec{u}$.

$|\vec{v}| = \sqrt{1^2 + 2^2} = \sqrt{5} \Rightarrow \vec{u} = \frac{1}{\sqrt{5}}\langle 1, 2 \rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

8. Compute $D_{\vec{u}} f(1,1)$.

$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u} = \langle -\frac{2}{3}, -\frac{1}{2} \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \frac{-2}{3\sqrt{5}} - \frac{1}{\sqrt{5}}$

$= \frac{-5}{3\sqrt{5}} = -\frac{\sqrt{5}}{3}$

9. What does $D_{\vec{u}} f(1,1)$ represent, in a geometric sense? What does it tell you about f?

The rate of change of f at $(1,1)$ in the direction of $\vec{u}$.

f is decreasing at point $(1,1)$ in the direction of $\vec{u}$.

10. Write the vector with x and y components matching those of $\vec{u}$ and z component $D_{\vec{u}}f(1,1)$. That is,

$$\langle u_1, u_2, D_{\vec{u}}f(1,1)\rangle$$

$\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{\sqrt{5}}{3} \right\rangle$

11. In a new input box, graph the vector you just found, starting from the point $(1, 1, f(1,1))$. What do you observe?

$R = \left(1 - \frac{1}{\sqrt{5}}, 1 - \frac{2}{\sqrt{5}}, f(1,1) + \frac{\sqrt{5}}{3}\right)$

Vector (P, R)

This vector gives the direction of f at point P.

Now, you can try this process without training wheels. Below is another function, point, and direction vector.

$$f(x,y) = \cos(x) - \frac{1}{2}y, P = \left(\frac{\pi}{2}, -2, f\left(\frac{\pi}{2}, -2\right)\right), \vec{d} = \langle 1, 1\rangle$$

12. Graph the surface, point, gradient vector (starting at the point), and directional derivative in the direction of $\vec{d}$ (starting at the point).

And here are some final questions to consider to help drive home the *meanings* of these new mathematical concepts.

13. Let $f(x,y) = 2x^2 + 5y$ and $\vec{u} = \langle 1, 0\rangle$. Find the directional derivative $D_{\vec{u}}f$.

14. A fly is flying around a room in which the temperature is given by $T(x,y,z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1,1,1)$ and realizes he's cold. In which direction should he fly to warm up most quickly? (Note: You can't graph this surface, as it is in 4-dimensional space.)

Homework C.4 - Directional Derivatives and the Gradient Vector

Prof. Leyda Almodóvar

Name: _____

1. Find the gradient of f. Evaluate the gradient at the point P. Then, find the rate of change of f at P in the direction of the vector $\vec{u}$.

   (a) $f(x,y) = x^2 \ln y$, $P = (3,1)$, $\vec{u} = -\frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}$

   $\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \boxed{2x \ln y \vec{i} + \left(\frac{x^2}{y}\right)\vec{j}}$

   $\nabla f(3,1) = 0\vec{i} + \left(\frac{9}{1}\right)\vec{j} = \boxed{9\vec{j}}$

   $D_u f(3,1) = \nabla f(3,1) \cdot \vec{u} = 9\vec{j} \cdot \left(-\frac{5}{13}\vec{i} + \frac{12}{13}\vec{j}\right) = 0 + \frac{108}{13} = \boxed{\frac{108}{13}}$

   (b) $f(x,y,z) = x^2yz - xyz^3$, $P = (2,-1,1)$, $\vec{u} = \langle 0, \frac{4}{5}, -\frac{3}{5}\rangle$

   $\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \boxed{(2xyz - yz^3)\vec{i} + (x^2z - xz^3)\vec{j} + (x^2y - 3xyz^2)\vec{k}}$

   $\nabla f(2,-1,1) = (-4+1)\vec{i} + (4-2)\vec{j} + (-4+6)\vec{k} = -3\vec{i} + 2\vec{j} + 2\vec{k} = \boxed{\langle -3, 2, 2\rangle}$

   $D_u f(2,-1,1) = \nabla f(-2,-1,1) \cdot \vec{u} = \langle -3,2,2\rangle \cdot \langle 0, \frac{4}{5}, -\frac{3}{5}\rangle = 0 + \frac{8}{5} - \frac{6}{5} = \boxed{\frac{2}{5}}$

2. Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

   (a) $f(x,y) = e^x \sin y$, $\left(0, \frac{\pi}{3}\right)$, $\vec{v} = \langle -6, 8\rangle$

   $\nabla f(x,y) = \langle e^x \sin y, e^x \cos y\rangle$,

   $\nabla f\left(0, \frac{\pi}{3}\right) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$,

a unit vector in the direction of $\vec{v}$ is $\vec{u} = \frac{1}{\sqrt{(-6)^2+8^2}}\langle -6, 8\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$,

$\Rightarrow \quad D_u f\left(0, \frac{\pi}{3}\right) = \nabla f\left(0, \frac{\pi}{3}\right) \cdot \vec{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \boxed{\frac{4-3\sqrt{3}}{10}}$

(b) $g(u, v) = u^2 e^{-v}$, $(3, 0)$, $\vec{v} = 3\vec{i} + 4\vec{j}$

$\nabla g(u, v) = (2ue^{-v})\vec{i} + (-u^2 e^{-v})\vec{j}$,

$\nabla g(3, 0) = 6\vec{i} - 9\vec{j}$,

a unit vector in the diretion of $\vec{v}$ is $\vec{u} = \frac{1}{\sqrt{3^2+4^2}}(3\vec{i} + 4\vec{j}) = \frac{1}{5}(3\vec{i} + 4\vec{j})$,

$\Rightarrow \quad D_u g(3, 0) = \nabla g(3, 0) \cdot \vec{u} = (6\vec{i} - 9\vec{j}) \cdot \frac{1}{5}(3\vec{i} + 4\vec{j}) = \frac{1}{5}(18 - 36) = \boxed{-\frac{18}{5}}$

3. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P = (2, 8)$ in the direction of $Q = (5, 4)$.

$\nabla f(x, y) = \left\langle \frac{1}{2}(xy)^{-\frac{1}{2}}(y), \frac{1}{2}(xy)^{-\frac{1}{2}}(x)\right\rangle = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}}\right\rangle$

$\nabla f(2, 8) = \left\langle 1, \frac{1}{4}\right\rangle$

The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8\rangle = \langle 3, -4\rangle$ is $\vec{u} = \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle$,

$\Rightarrow \quad D_u f(2, 8) = \nabla f(2, 8) \cdot \vec{u} = \left\langle 1, \frac{1}{4}\right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5}\right\rangle = \boxed{\frac{2}{5}}$

4. Find the maximum rate of change of f at the given point and the direction in which it occurs.

(a) $f(s, t) = te^{st}$, $(0, 2)$

$\nabla f(s, t) = \langle te^{st}(t), te^{st}(s) + e^{st}(1)\rangle = \langle t^2 e^{st}, (st+1)e^{st}\rangle$

$\nabla f(0, 2) = \langle 4, 1\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(0, 2)| = \sqrt{16+1} = \boxed{\sqrt{17}}$

(b) $f(x, y) = \sin(xy)$, $(1, 0)$

$\nabla f(x, y) = \langle y\cos(xy), x\cos(xy)\rangle$

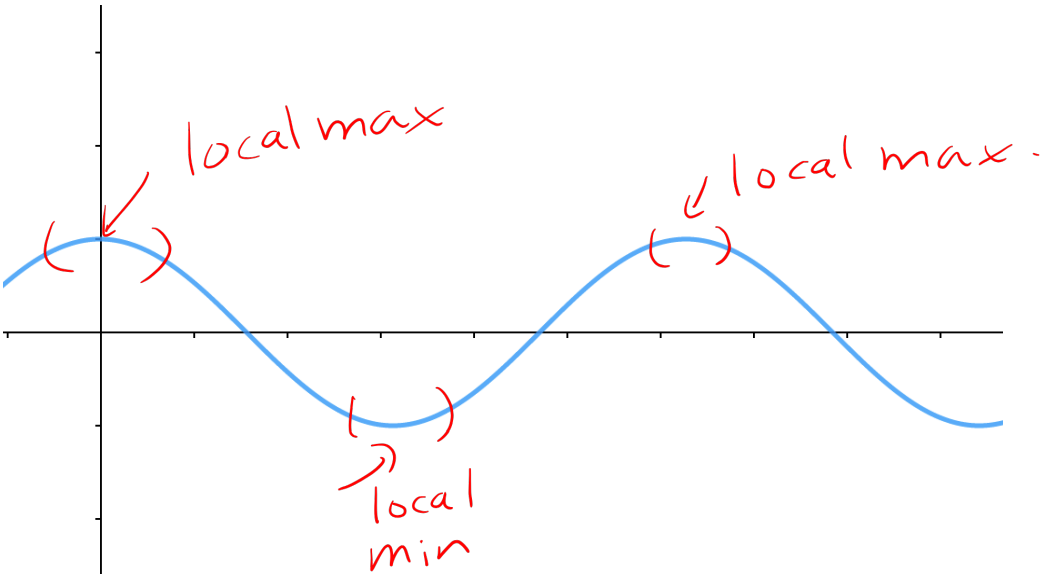
$\nabla f(0, 1) = \langle 0, 1\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(0, 1)| = \boxed{1}$

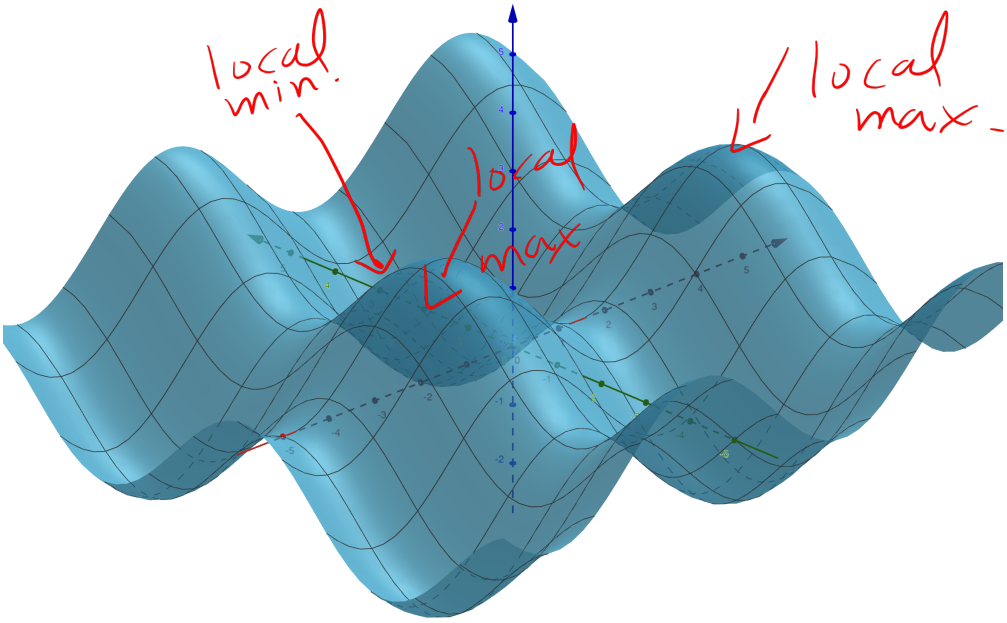
C.5 Extrema & Optimization

In Calculus I we find maximums and minimums of single-variable functions by taking their derivatives, solving for critical points (where the derivative equals zero or is undefined), and applying a second derivative test to determine if the function is concave up or down to find a minimum or maximum, respectively. We seek to build the analog of this process for functions of two variables.

Definition. For a function f (of any number of variables),

- f has a **local maximum** at the point P_0 if $f(P_0) \geq f(P)$ for all points P *near* P_0.
- f has a **local minimum** at the point P_0 if $f(P_0) \leq f(P)$ for all points P *near* P_0.





Lab 7: Finding Extrema

1. Open up a 3D graphing page on *Geogebra* and graph the surface
$$f(x,y) = -x^2 - y^2 + 4x + 6y - 9$$
2. In a new input box, type the point $P = (2, 3, f(2,3))$. What do you observe about this point? What feature does it represent of the surface? It's the highest point on the surface (maximum)
3. Find f_x and f_y, and then find $\nabla f(2,3)$. What do you observe?
$f_x = -2x + 4$ $\quad \nabla f(2,3) = \langle -2(2)+4, -2(3)+6 \rangle$
$f_y = -2y + 6$ $\quad = \langle 0, 0 \rangle$
4. In the context of what we've learned about the meaning of the gradient vector, what does your previous answer indicate? What is the gradient vector of f at the point $(2,3)$ "telling you to do"?
There is no change at P.
It's telling us to do nothing.

Let's repeat that exercise in a slightly different scenario.

5. Graph the surface
$$g(x,y) = x^2 + y^2 - 4x - 6y + 17$$
6. In a new input box, type the point $Q = (2, 3, g(2,3))$. What do you observe about this point? What feature does it represent of the surface? It's the lowest point on the surface (minimum)
7. Find g_x and g_y, and then find $\nabla g(2,3)$. What do you observe?
$g_x = 2x - 4$ $\quad \nabla g(2,3) = \langle 2(2)-4, 2(3)-6 \rangle$
$g_y = 2y - 6$ $\quad = \langle 0, 0 \rangle$
8. In the context of what we've learned about the meaning of the gradient vector, what does your previous answer indicate? What is the gradient vector of g at the point $(2,3)$ "telling you to do"?
There is no change at Q.
It's telling us to do nothing.

Okay, so we need way to draw a distinction between these two cases.

9. Find f_{xx} and f_{yy} of $f(x,y) = -x^2 - y^2 + 4x + 6y - 9$.

$f_{xx} = -2 < 0$

$f_{yy} = -2 < 0$

10. Find $f_{xx}(2,3)$ and $f_{yy}(2,3)$ (hint: this might be zero effort).

11. What do you think your previous answer tells you about the surface at the point $(2,3)$? Can it help you determine what kind of point $(2,3,f(2,3))$ is?

It seems to suggest that P is a local max.

12. Repeat the previous three questions for $g(x,y) = x^2 + y^2 - 4x - 6y + 17$.

$g_{xx} = 2$ $\quad g_{xx}(2,3) = 2 > 0$

$g_{yy} = 2$ $\quad g_{yy}(2,3) = 2 > 0$

It seems to suggest that Q is a local min.

It seems like you have discovered something that you can "reverse-engineer" in order to find extrema of surfaces. Let's try this on one more example.

13. Consider, but DO NOT GRAPH, the function:

$$h(x, y) = -x^2 + 4x + y^2 - 6y + 9$$

14. Find ∇h.

$h_x = -2x + 4$ $\quad \nabla h = \langle -2x+4, 2y-6 \rangle$

$h_y = 2y - 6$

15. Find a point (a, b) such that $\nabla h(a, b) = \vec{0}$.

$-2x+4=0$ and $2y-6=0$

$x=2$ $\quad y=3$ $\quad \nabla h(2,3) = \vec{0}$

16. Find $h_{xx}(a, b)$ and $h_{yy}(a, b)$.

$h_{xx}(2,3) = -2 < 0$

$h_{yy}(2,3) = 2 > 0$

17. Hmm, that's different. What do you think your previous answer tells you about the surface at the point (a, b)? Do you think (a, b) is an extremum of h?

No

18. Graph the surface h and the point you found. What do you observe?

$(2,3)$ is not an extreme value of h.

Lab 7 Follow-Up Notes

In single-variable calculus, critical points occur when the derivative equals zero or is undefined. We have to work with **partial** derivatives when we are dealing with functions of two variables. What have we studied that could encompass both partial derivatives?

Definition. A **critical point** of $f(x, y)$ is a point (a, b) where:

- $f_x(a,b)=0$ or $f_x(a,b)$ D.N.E. AND
- $f_y(a,b)=0$ or $f_y(a,b)$ D.N.E.

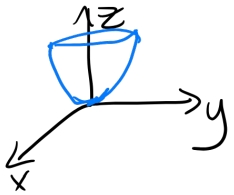
} $\nabla f = \vec{0}$ or D.N.E.

Example. Find the critical points of $f(x, y) = x^2 + y^2$.

$\nabla f = \langle 2x, 2y \rangle = \vec{0} \Rightarrow 2x=0$ and $2y=0$

$\Rightarrow (0,0)$ is the only critical point

What does $f(x, y) = x^2 + y^2$ look like? Why does your answer make sense?



Because $(0,0)$ is the lowest point on the surface. There's no maximum.

Example. Find the critical points of $g(x, y) = 8y^3+12x^2-24xy$. Observe the graph of g on Geogebra and WolframAlpha to visualize maximums/minimums.

$\nabla g = \langle 24x-24y, 24y^2-24x \rangle = \langle 0,0 \rangle$

$24x-24y=0$ and $24y^2-24x=0$

$x=y$ $\qquad x=y^2$

$\Rightarrow y=y^2 \Rightarrow 0=y^2-y$

$0=y(y-1)$

$y=0, y=1$

critical points: $(0,0)$, $(1,1)$

$\Rightarrow (0,0)$ is not a local min/max

$(1,1)$ is a local min.

Fact. If f has a local maximum/minimum at P_0, then P_0 is a critical point. The converse is not necessarily true, as we saw in the lab.

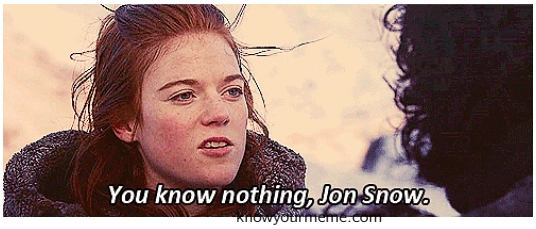
(a, b) is a critical point

Second Derivative Test for Functions of Two Variables:
Suppose (a, b) is a point where $\nabla f(a, b) = \vec{0}$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

- If $D > 0$ and $f_{xx}(a, b) > 0$ then f has a **local minimum** at (a, b).
- If $D > 0$ and $f_{xx}(a, b) < 0$ then f has a **local maximum** at (a, b).
- If $D < 0$ then f has a **saddle point** at (a, b).
- If $D = 0$ we know nothing.

not a min or max



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Worksheet: Second Derivative Test Practice

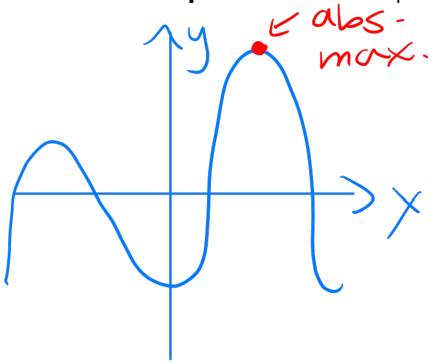
Optimization Strategies

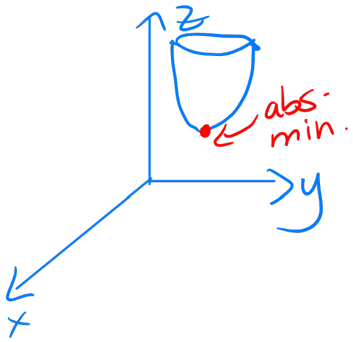
Sometimes we seek to find local maximums and minimums, and sometimes we seek to find the absolute largest or smallest values of a function.

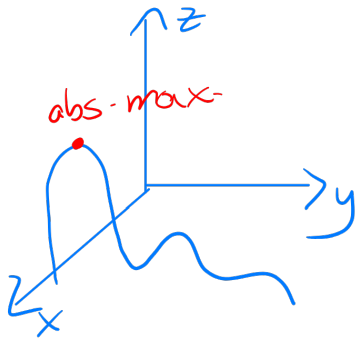
Definition. For a function f defined on a region R,

- f has a **global** or **absolute maximum** on R at the point P_0 if $f(P_0) \geq f(P)$ for *all* points P in R.
- f has a **global** or **absolute minimum** on R at the point P_0 if $f(P_0) \leq f(P)$ for *all* points P in R.

Example. A few examples in two and three dimensions:







Note. If the region over which we are working is specific or limited, the process of finding global extrema is called **constrained optimization.** If the region is the entire space of input variables (read: in Calculus I the entire x-axis; in Calculus III the entire xy-plane), then the process is called **unconstrained optimization**. So far, we have been, by default, in unconstrained optimization mode.

Process for Unconstrained Optimization:

1. Find critical points.
2. Use second derivative test to locate local maximums/minimums.
3. Compare the discovered local maximums/minimums to one another and to the "end behavior" of the function (the graph is helpful).

Example. Determine all global extrema of $f(x,y) = x^2 + 2y^2$.

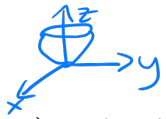
$\nabla f(x,y) = \langle 2x, 4y \rangle = \langle 0,0 \rangle \Rightarrow x=0, y=0 \Rightarrow (0,0)$ is the only C.P.

$f_{xx} = 2, f_{yy} = 4, f_{xy} = 0$

$D = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 2 \cdot 4 - 0^2 = 8 > 0$ and $f_{xx}(0,0) > 0$

$\Rightarrow$ local min. at $(0,0)$

shortest



Looking at graph we confirm $(0,0)$ yields a global min.

Example. Find the distance from the point $(1,2,-1)$ to the plane $x - y + z = 3$.

Let d be the distance from the point $(1,2,-1)$ to a point (x,y,z) on $x-y+z=3$.

Then, $d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$. Goal: Minimize d.

Consider $d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$ [It's equivalent to minimize d^2 or d]

$z = -x+y+3 \Rightarrow d^2 = f(x,y) = (x-1)^2 + (y-2)^2 + (-x+y+4)^2$

$f_x = 2(x-1) + 2(-x+y+4)(-1) = 4x - 2y - 10 = 0$

$f_y = 2(y-2) + 2(-x+y+4) = -2x + 4y + 4 = 0$

$\Rightarrow \quad 8x - 4y - 20 = 0$

$+(-2x + 4y + 4 = 0)$

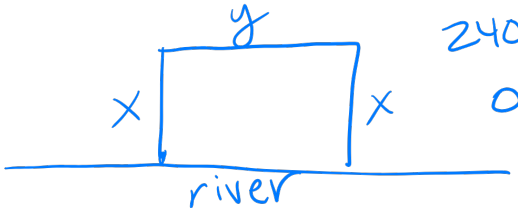
$6x - 16 = 0 \Rightarrow x = \frac{16}{6} = \frac{8}{3}$

$4(\frac{8}{3}) - 2y - 10 = 0 \Rightarrow \frac{32}{3} - 10 = 2y \Rightarrow \frac{2}{3} = 2y \Rightarrow y = 1/3$

$\Rightarrow (\frac{8}{3}, \frac{1}{3})$ is the only C.P. Since the shortest distance exists, it must be $f(\frac{8}{3}, 1/3)$.

When considering optimization in a specific region in single-variable calculus we are working over a particular portion of the x-axis.

Example. Examples for single-variable functions:



2400 ft of fence

$0 \le x \le 1200$

Recall:

1) Find the C.P.s
2) Evaluate f at C.P.s
3) Evaluate f at endpoints of interval

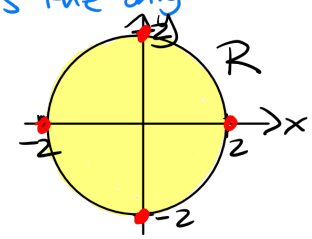
Process for Constrained Optimization:

1. Find critical points of f inside the region and f's value at each of them.
2. Find the extreme values of f on the boundary of the region (note: this requires a boundary).
3. Compare all values from steps 1 and 2; the largest is the absolute maximum, the smallest is the absolute minimum.

Example. Consider the function $f(x,y) = x^2 + 2y^2$. Find all global extrema of f on the region $R = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

Find C.P.s: $\nabla f(x,y) = \langle 2x, 4y \rangle = \langle 0,0 \rangle \Rightarrow (0,0)$ is the only C.P.

$f(0,0) = 0 \Rightarrow (0,0,0)$ → (only the circle)



Check the boundary: $x^2 + y^2 = 4$

$y^2 = 4 - x^2 \Rightarrow g(x) = x^2 + 2(4 - x^2) = -x^2 + 8$

$g'(x) = -2x = 0 \Rightarrow x = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$ → $(0,-2)$, $(0,2)$

$f(0,-2) = 0 + 2(-2)^2 = 8$ } $(0,-2,8)$

$f(0,2) = 0 + 2(2)^2 = 8$ } $(0,2,8)$

$x^2 = 4 - y^2 \Rightarrow h(y) = 4 - y^2 + 2y^2 = y^2 + 4$

$\Rightarrow h'(y) = 2y = 0 \Rightarrow y = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ → $(-2,0)$, $(2,0)$

$f(-2,0) = (-2)^2 + 2(0)^2 = 4$ } $(-2,0,4)$

$f(2,0) = 2^2 + 2(0)^2 = 4$ } $(2,0,4)$

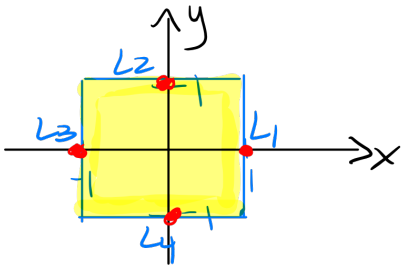
We compare the five points and find that $(0,0,0)$ is the global minimum and $(0,-2,8)$ and $(0,2,8)$ are the global max. in R.

The previous example is a *constrained* optimization problem. Most real world optimization problems have constraints. Examples: Materials for building, budgets, work force available, time, etc.

same function, different region

Example. Find all global extrema of $g(x,y) = x^2 + 2y^2$ on the region $R = \{(x,y) \mid |x| \le 1, |y| \le 1\}$. Try to do so algebraically before consulting a graph.

Find C.Ps. $\nabla g(x,y) = \langle 2x, 4y \rangle = \langle 0,0 \rangle \Rightarrow (0,0)$ is the only C.P. $\Rightarrow (0,0,0)$



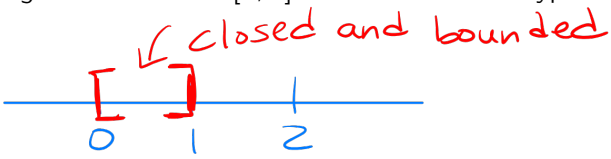
Check the boundary:

- On L_1: $x = 1$, $g(1,y) = 1 + 2y^2$
  $\Rightarrow g' = 4y = 0 \Rightarrow y = 0$
  $g(1,0) = 1^2 + 2(0)^2 = 1 \Rightarrow (1,0,1)$
- On L_2: $y = 1$, $g(x,1) = x^2 + 2 \Rightarrow g' = 2x = 0 \Rightarrow x = 0$
  $g(0,1) = 0^2 + 2(1)^2 = 2 \Rightarrow (0,1,2)$
- On L_3: $x = -1$, $g(-1,y) = 1 + 2y^2 \Rightarrow g' = 4y = 0 \Rightarrow y = 0$
  $g(-1,0) = (-1)^2 + 2(0)^2 = 1 \Rightarrow (-1,0,1)$
- On L_4: $y = -1$, $g(x,-1) = x^2 + 2 \Rightarrow g' = 2x = 0 \Rightarrow x = 0$
  $g(0,-1) = 0^2 + 2(-1)^2 = 2 \Rightarrow (0,-1,2)$

We find that $(0,0,0)$ is the global min and $(0,1,2)$ and $(0,-1,2)$ are the global max in R.

Recall. The **Extreme Value Theorem** from single-variable calculus provides a guarantee about extrema over a region of the form $[a, b]$.

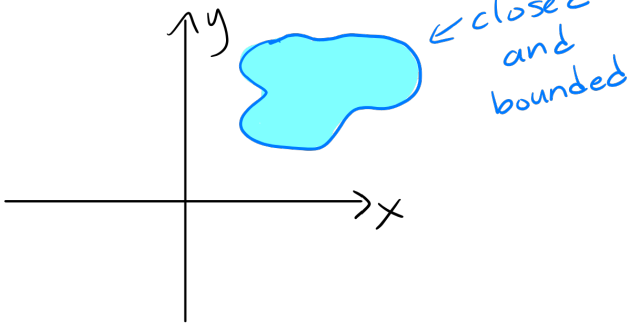
A region of the form $[a, b]$ on the x-axis is a type of region called *closed* and *bounded*.

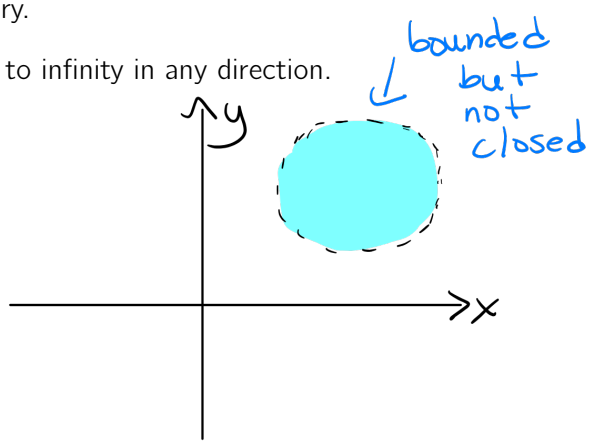


Definition. A **closed region** is one which contains its boundary.

Definition. A **bounded region** is one which does not stretch to infinity in any direction.

Example. Examples of each/both:





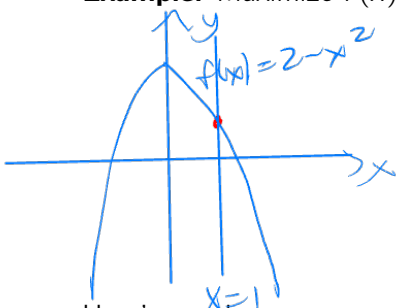
We have an analogous theorem to the one from single variable calculus.

Theorem (Extreme Value Theorem for Multivariable Functions)**.** If f is a continuous function on a *closed and bounded region* R, then f has a global maximum in R and a global minimum in R.

Lagrange Multipliers

Consider a different example of constrained optimization. This one is in two dimensions.

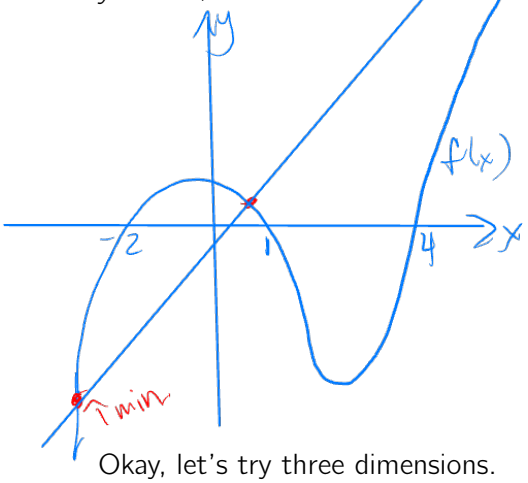
Example. Maximize $f(x) = 2 - x^2$ subject to the constraint $x - 1 = 0$.



There's only one point to consider

Here's another:

Example. Maximize $f(x) = 2(x+2)(x-1)(x-4) = 2x^3 - 6x^2 - 12x + 16$ subject to the constraint $y = 3x + 1$.



Note: The min/max were obtained by looking at the intersection of the curve with the constraint

Okay, let's try three dimensions.

Example. Maximize $f(x, y) = 2 - x^2 - 2y^2$ subject to the constraint $x^2 + y^2 = 1$. Then, sketch a contour diagram of f and add the constraint function to that graph.

CPs: $\nabla f(x,y) = \langle -2x, -4y \rangle = \langle 0, 0 \rangle \Rightarrow (0,0)$, $f(0,0) = 2 \Rightarrow (0,0,2)$ ↙ max

Boundary: $y^2 = 1 - x^2 \Rightarrow g(x) = 2 - x^2 - 2(1-x^2) = x^2 \Rightarrow g'(x) = 2x = 0 \Rightarrow x = 0$

$y^2 = 1 \rightarrow (0,-1)$, $(0,1)$ $\quad f(0,\pm 1) = 2 - 0^2 - 2(\pm 1)^2 = 0 \Rightarrow (0,-1,0)$, $(0,1,0)$ ↙ min

$x^2 = 1 - y^2 \Rightarrow h(y) = 2 - (1-y^2) - 2y^2 = 1 - y^2 \Rightarrow h'(y) = -2y = 0 \Rightarrow y = 0$

$x^2 = 1 \rightarrow (-1,0)$, $(1,0)$ $\quad f(\pm 1, 0) = 2 - (\pm 1)^2 - 2 \cdot 0^2 = 1 \Rightarrow (1,0,1)$, $(-1,0,1)$

$z = 0$: $0 = 2 - x^2 - 2y^2$, $x^2 + 2y^2 = 2$

$z = 1$: $1 = 2 - x^2 - 2y^2$, $x^2 + 2y^2 = 1$

So far we have been successful in constrained examples by using substitution. However, it has been difficult at times and could be even worse! Let's try to discover another tool.

Consider the previous contour diagram example:

- The extrema occurred when the constraint function and a contour of the function $f(x, y)$ intersected.
- This implies that the gradient vectors of f at the location of the maxima would be perpendicular not only to the contours of f (as they always are), but also to the constraint function.
- So, gradient vectors to the constraint and surface functions are parallel at any extreme point P_0.
- Formally: $\nabla(f(P_0)) = \lambda \cdot \nabla(g(P_0))$ (along with $g(P_0) = 1$ in that example).

extreme point, constraint, function, constant, parallel vectors are scalar multiples of each other

$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$ (if P_0 is extreme point)

Optimizing f Subject to a Constraint g = k :
If f is smooth and has a maximum or minimum subject to the constraint $g = k$ at point P_0, then one of the following must be true:

In example $x^2y^2 = 1$, $g = 1$

- P_0 satisfies: $\nabla f = \lambda \cdot \nabla g$ and $g = k$
- $\nabla(g(P_0)) = \vec{0}$

To find maxima and minima compare the values of f at the points P_0 that satisfy those conditions. The scalar λ is called the **Lagrange multiplier**, so this process is called the **Lagrange Multiplier Method**.

Example. Use the Lagrange Multiplier Method to find maxima and minima of $f(x,y) = x + y$ subject to the constraint $x^2 + y^2 = 4$.

$\nabla f(x,y) = \langle 1, 1 \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$ (g)

$\nabla g(x,y) = \langle 2x, 2y \rangle = \langle 0, 0 \rangle \Rightarrow (0,0) \Rightarrow (0,0,0)$

and

$\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$

$\langle 1, 1 \rangle = \lambda \cdot \langle 2x, 2y \rangle$

$1 = \lambda \cdot 2x$ and $1 = \lambda \cdot 2y$ (note $\lambda \neq 0, x \neq 0, y \neq 0$)

$\frac{1}{2x} = \lambda$ and $\frac{1}{2y} = \lambda \Rightarrow \frac{1}{2x} = \frac{1}{2y} \Rightarrow x = y$

$\Rightarrow x^2 + x^2 = 4 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2$ → $x = \sqrt{2}$ → $(\sqrt{2}, \sqrt{2})$; $x = -\sqrt{2}$ → $(-\sqrt{2}, -\sqrt{2})$

$\Rightarrow (\sqrt{2}, \sqrt{2}, 2\sqrt{2})$ max; $(-\sqrt{2}, -\sqrt{2}, -2\sqrt{2})$ min

Example. Use the Lagrange Multiplier Method to find maxima and minima of $g(x,y) = x^2 + y^2$ subject to $xy = 3$.

$\nabla h(x,y) = \vec{0} \Rightarrow \langle y, x \rangle = \langle 0, 0 \rangle \Rightarrow x = y = 0$

$(0,0,0)$ min.

(h)

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$\nabla h(x,y) = \langle y, x \rangle$

$\nabla g(x,y) = \lambda \cdot \nabla h(x,y)$

$\langle 2x, 2y \rangle = \lambda \cdot \langle y, x \rangle$

$2x = \lambda \cdot y$ and $2y = \lambda \cdot x$

If $\lambda = 0$, $x = 0$ and $y = 0$

If $x \neq 0$ $\frac{2x}{y} = \lambda$ and $\frac{2y}{x} = \lambda \Rightarrow \frac{2x}{y} = \frac{2y}{x} \Rightarrow 2x^2 = 2y^2 \Rightarrow y = \pm x$

If $y = x$, $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$ → $(\sqrt{3}, \sqrt{3})$; $(-\sqrt{3}, -\sqrt{3})$ (g) → $(\sqrt{3}, \sqrt{3}, 6)$; $(-\sqrt{3}, -\sqrt{3}, 6)$ max.

If $y = -x$, $-x^2 = 3$ (imaginary numbers)

Note. If a constraint is given as $g(x,y) \leq k$, a closed and bounded region, then use the Lagrange Multiplier method for $g(x,y) = k$, find all critical points of f inside the region, and compare values at all of those points.

Example. Use the Lagrange Multiplier Method to find maxima and minima of $h(x,y) = 4x^2+y^2-4xy$ on the region $x^2+y^2 \leq 1$.

g, $x^2+y^2=1$

$\nabla h(x,y) = \langle 8x-4y, 2y-4x \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

Lagrange: $\nabla g = \vec{0} \Rightarrow x=y=0 \Rightarrow (0,0,0)$

$\nabla h = \lambda \cdot \nabla g$

$\langle 8x-4y, 2y-4x \rangle = \lambda \cdot \langle 2x, 2y \rangle$

$8x-4y = \lambda \cdot 2x$ and $2y-4x = \lambda \cdot 2y$

$\frac{8x-4y}{2x} = \lambda$ and $\frac{2y-4x}{2y} = \lambda \Rightarrow \frac{8x-4y}{2x} = \frac{2y-4x}{2y} \Rightarrow 8xy-4y^2 = 2xy-4x^2$

$4xy-2y^2 = xy-2x^2$

$3xy-2y^2+2x^2=0 \Rightarrow 2x-y=0$ or $x+2y=0$

$(2x-y)(x+2y)=0$ $\quad y=2x$ or $x=-2y$

If $x^2+y^2=1$, $x^2+(2x)^2=1 \Rightarrow 5x^2=1 \Rightarrow x = \pm\frac{1}{\sqrt{5}} \Rightarrow y = \pm\frac{2}{\sqrt{5}}$

$\Rightarrow \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$ Note $h\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = h\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) = 0$

$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ and $\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$

$(-2y)^2+y^2=1 \Rightarrow 5y^2=1 \Rightarrow y = \pm\frac{1}{\sqrt{5}} \Rightarrow x = \pm\frac{2}{\sqrt{5}}$

$\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ Note $h\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) = h\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = 5$

$\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 5\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 5\right)$ ← max.

Inside disk: $\nabla h = \vec{0} \Rightarrow 8x-4y=0$ and $2y-4x=0$

$8x=4y$ $\quad 2y=4x$

$2x=y$ $\quad y=2x$

$h(x,2x) = 4x^2+(2x)^2-4x(2x) = 4x^2+4x^2-8x^2 = 0$

$(x,2x,0)$ ← CPs

$(x,2x,0)$ are minima $\left[\frac{-1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}, \quad \frac{-2}{\sqrt{5}} < y < \frac{2}{\sqrt{5}}\right]$

and $\left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 5\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 5\right)$ are maxima

Homework C.5 - Extrema and Optimization

Prof. Leyda Almodóvar

Name: _____

1. Suppose $(1,1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f?

   (a) $f_{xx}(1,1) = 4,\ f_{xy}(1,1) = 1,\ f_{yy}(1,1) = 2$

   $D(1,1) = f_{xx}(1,1)f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - 1 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

   (b) $f_{xx}(1,1) = 4,\ f_{xy}(1,1) = 3,\ f_{yy}(1,1) = 2$

   $D(1,1) = f_{xx}(1,1)f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - 3^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

2. Find the local maximum and minimum values and saddle point(s) of the function.

   (a) $f(x,y) = xy - 2x - 2y - x^2 - y^2$

   $f_x = y - 2 - 2x, \quad f_y = x - 2 - 2y, \quad f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1.$

   $f_x = 0$ implies $y = 2x+2$, and substitution into $f_y = 0$ gives $x-2-2(2x+2) = 0$, which implies $-3x = 6 \Rightarrow x = -2$. Then, $y = -2$ and the only critical point is $(-2,2)$.

   $D(-2,-2) = f_{xx}(-2,-2)f_{yy}(-2,-2) - [f_{xy}(-2,-2)]^2 = (-2)(-2) - 1^2 = 3$, and since $D(-2,-2) = 3 > 0$ and $f_{xx}(-2,-2) = -2 < 0$, $f(-2,-2) = 4$ is a local maximum by the Second Derivatives Test.

   (b) $f(x,y) = y(e^x - 1)$

   $f_x = ye^x, \quad f_y = e^x - 1, \quad f_{xx} = ye^x, \quad f_{yy} = e^x, \quad f_{xy} = e^x.$

   Because e^x is never zero, $f_x = 0$ only when $y = 0$, and $f_y = 0$ when $e^x = 1$, that is, $x = 0$. The only critical point is $(0,0)$.

   $D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - (e^x)^2 = -e^{2x}$, and since $D(0,0) = -1 < 0$, $(0,0)$ is a saddle point.

(c) $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

$f_x = 3x^2 + 6x + 9, \quad f_y = 3y^2 - 6y, \quad f_{xx} = 6x - 6, \quad f_{yy} = 6y - 6, \quad f_{xy} = 0$

Then $f_x = 0$ implies $3(x+1)(x-3) = 0 \Rightarrow x = -1$ or $x = 3$, and $f_y = 0$ implies $3y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$.

The critical points are $(-1,0)$, $(-1,2)$, $(3,0)$, and $(3,2)$.

$D(-1,2) = (-12)(6) - (0)^2 = -72 < 0, \quad D(3,0) = (12)(-6) - (0)^2 = -72 < 0,$ so $(-1,2)$ and $(3,0)$ are saddle points.

$D(-1,0) = (-12)(-6) - (0)^2 = 72 > 0$ and $f_{xx}(-1,0) = -12 < 0$, so $f(-1,0) = 5$ is a local maximum.

$D(3,2) = (12)(6) - (0)^2 = 72 > 0$ and $f_{xx}(3,2) = 12 > 0$, so $f(3,2) = -31$ is a local minimum.

(d) $f(x,y) = e^x \cos y$

$f_x = e^x \cos y$, $f_y = -e^x \sin y$. Now, $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer. But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.

(e) $f(x,y) = (x^2 + y^2)e^{-x}$

$f_x = (x^2 + y^2)(-e^{-x}) + e^{-x}(2x) = (2x - x^2 - y^2)e^{-x},$

$f_y = 2ye^{-x},$

$f_{xx} = (2x - x^2 - y^2)(-e^{-x}) + e^{-x}(2 - 2x) = (x^2 + y^2 - 4x + 2)e^{-x},$

$f_{yy} = 2e^{-x}$, $f_{xy} = -2ye^{-x}$

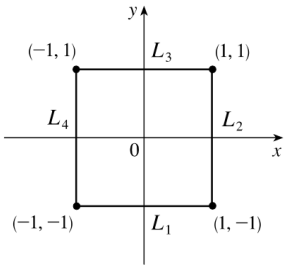
Then, $f_y = 0$ implies $y = 0$ and substituting into $f_x = 0$ gives $(2x - x^2)e^{-x} = 0$ implies $x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$, so the critical points are $(0,0)$ and $(2,0)$.

$D(0,0) = (2)(2) - (0)^2 = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $f(0,0) = 0$ is a local minimum.

$D(2,0) = (-2e^{-2})(2e^{-2}) - (0)^2 = -4e^{-4} < 0$, so $(2,0)$ is a saddle point.

3. Find the absolute maximum and minimum values of f on the region R.

(a) $f(x,y) = x^2 + y^2 + x^2y + 4$, $R = \{(x,y) : |x| \leq 1, |y| \leq 1\}$



$f_x(x,y) = 2x + 2xy$, $f_y(x,y) = 2y + x^2$, and setting $f_x = f_y = 0$ implies $2x(y+1) = 0$ and $2y + x^2 = 0$. If $x = 0$, $2y = 0$ implies $y = 0$. If $y = -1$, $x^2 = 2$, but $x = \pm\sqrt{2}$ does not satisfy $|x| \leq 1$. Hence $(0,0)$ is the only critical point in R, with $f(0,0) = 4$.

We will find critical points inside the lines L_1, L_2, L_3, and L_4 and we also need to check the extremes of the intervals $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. That is, the points $(-1,-1)$, $(-1,1)$, $(1,1)$ and $(-1,1)$.
On L_1: $y = -1$, $f(x,-1) = 5$, a constant.

On L_2: $x = 1$, $g(y) = f(1,y) = y^2+y+5$, then $g'(y) = 2y+1 = 0$ when $y = -\frac{1}{2}$. Then $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$. We can test the endpoints $(1,1)$ and $(1,-1)$ with this function since they lie on L_2. $f(1,1) = 7$ and $f(1,-1) = 5$.

On L_3: $h(x) = f(x,1) = 2x^2 + 5$, then $h'(x) = 4x = 0$ when $x = 0$ and $f(0,1) = 5$. Note that the endpoint $(-1,1)$ lies on this line and $h(-1,1) = 7$.

On L_4: $f(-1,y) = y^2+y+5$, then we obtain $\left(-1, -\frac{1}{2}\right)$ as a critical point. Then $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$. The endpoint $(-1,-1)$ lies on this line and $f(-1,1) = 7$.

This the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on r is attained at $(0,0)$ with $f(0,0) = 4$.

(b) $f(x,y) = 2x^3 + y^4$, $R = \{(x,y) : x^2 + y^2 \leq 1\}$

$f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. So, $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $(0,0)$, where $f(0,0) = 0$. Now, on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$, so let $g(x) = f(x,y) = 2x^3+(1-x^2)^2 = x^4+2x^3-2x^2+1$, $-1 \leq x \leq 1$. Then, $g'(x) = 4x^3+6x^2-4x = 0$, which implies $x = 0$, $x = -2$, or $x = \frac{1}{2}$.

$f(0,\pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2,-3)$ is not in R.

$x^2 = 1 - y^2$, then $h(y) = f(x,y) = 2\left(\sqrt{1-y^2}\right)^3 + y^4$.

Then, $h'(y) = 3\sqrt{1-y^2}(-2y) + 4y^3 = 0$ when $2y(-3\sqrt{1-y^2} + 2y^2) = 0$, which implies $y = 0$, $2y^2 = 3\sqrt{1-y^2}$. Squaring both sides of the equation we obtain $4y^4 = 9(1 - y^f 2)$ or $4y^4 + 9y^2 - 9 = 0$. We can solve this equation as a quadratic by substituting $X = y^2$. We find $X = \frac{3}{4}$ and $X = -3$. This implies $y^2 = \frac{3}{4}$, and $y^2 = -3$. The first equation yields $y = \pm\frac{\sqrt{3}}{2}$, which are not in R and the second equation yields imaginary numbers.

If $y = 0$, then $x = \pm 1$ and $f(-1, 0) = g(-1) = -2$ and $f(1, 0) = g(1) = 2$.

Thus, the absolute maximum and minimum of f on R are $f(1, 0) = 2$ and $f(-1, 0) = -2$.

4. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 1, 1)$.

The distance d from a point on the plane to the point $(0, 1, 1)$ is $d = \sqrt{x^2 + (y-1)^2 + (z-1)^2}$, where $z = 2 - \frac{1}{3}x + \frac{2}{3}y$. We can minimize $d^2 = x^2 + (y-1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2$, so
$f_x(x, y) = 2x + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$ and
$f_y(x, y) = 2(y-1) + 2\left(1 - \frac{1}{3}x + \frac{2}{3}y\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$

Solving $\frac{20}{9}x - \frac{4}{9}y - \frac{2}{3} = 0$ and $-\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3} = 0$ simultaneously gives $x = \frac{5}{14}$ and $y = \frac{2}{7}$, so the only critical point is $\left(\frac{5}{14}, \frac{2}{7}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(0, 1, 1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

5. Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

   (a) $f(x, y) = 3x + y; x^2 + y^2 = 10$

   $\nabla f = \lambda \nabla g \Rightarrow \langle 3, 1\rangle = \langle 2\lambda x, 2\lambda y\rangle$, so $3 = 2\lambda x$, $1 = 2\lambda y$, and $x^2 + y^2 = 10$. From the first two equation we have $\frac{3}{2x} = \lambda = \frac{1}{2y} \Rightarrow x = 3y$ (note that the first two equations imply $x \neq 0$ and $y \neq 0$) and substitution into the third equation gives $9y^2 + y^2 = 10 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Then f has possible extreme values at the points $(3, 1)$ and $(-3, -1)$.

   We compute $f(3, 1) = 10$, and $f(-3, -1) = -10$, so the maximum value of f on $x^2 + y^2 = 10$ is $f(3, 1) = 10$ and the minimum value is $f(-3, -1) = -10$.

   (b) $f(x, y) = xe^y$; $x^2 + y^2 = 2$

   $\nabla f = \lambda \nabla g \Rightarrow \langle e^y, xe^y\rangle = \langle 2\lambda x, 2\lambda y\rangle$, so $e^y = 2\lambda x$, $xe^y = 2\lambda y$, and $x^2 + y^2 = 2$. First note that from the first equation $x \neq 0$. If $y = 0$, the second equation

implies $x = 0$, so $y \neq 0$. Then from the first two equations we have $\frac{e}{2x} = \lambda = \frac{xe^y}{2y}$ $\Rightarrow$ $2ye^y = 2x^2e^y$ $\Rightarrow$ $y = x^2$, and substituting into the third equation gives $x^2 + (x^2)^2 = 2 \Rightarrow x^4 + x^2 - 2 = 0 \Rightarrow (x^2 + 2)(x^2 - 1) = 0 \Rightarrow x = \pm 1$. From $y = x^2$ we have $y = 1$, so f has possible extreme values at $(\pm 1, 1)$. Evaluating f at these points, we see that the maximum value is $f(1, 1) = e$ and the minimum is $f(-1, 1) = -e$.

Worksheet - Multivariate Integration Practice

Compute the following integrals.

1. $\displaystyle\int_1^3 \int_0^1 (1+4xy)\,dx\,dy$

$$\int_1^3 \int_0^1 (1+4xy)\,dx\,dy = \int_1^3 \Big[x+2x^2y\Big]_0^1 dy = \int_1^3 (1+2y)\,dy = \Big[y+y^2\Big]_1^3 dy = (3+9)-(1+1) = \boxed{10}$$

2. $\displaystyle\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,dx\,dy$

$$\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y}\,dx\,dy = \int_0^{\ln 2} \Big[\frac{e^{2x-y}}{2}\Big]_0^{\ln 5} dy = \int_0^{\ln 2} \Big(\frac{e^{2\ln 5-y}}{2} - \frac{e^{-y}}{2}\Big)\,dy$$

$$= \int_0^{\ln 2} \Big(\frac{e^{\ln 25-y}}{2} - \frac{e^{-y}}{2}\Big)\,dy$$

$$= \Big[-\frac{e^{\ln 25-y}}{2} + \frac{e^{-y}}{2}\Big]_0^{\ln 2} = \Big(-\frac{e^{\ln 25-\ln 2}}{2} + \frac{e^{-\ln 2}}{2}\Big) - \Big(-\frac{e^{\ln 25-0}}{2} + \frac{e^0}{2}\Big)$$

$$= -\frac{e^{\ln \frac{25}{2}}}{2} + \frac{e^{\ln \frac{1}{2}}}{2} + \frac{25}{2} - \frac{1}{2} = -\frac{25}{4} + \frac{1}{4} + \frac{24}{2} = -\frac{24}{4} + 12 = \boxed{6}$$

3. $\displaystyle\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dx\,dy$

$$\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dx\,dy = \int_0^1 \int_{y^2}^{1+y^2} \frac{y}{2}\sqrt{u}\,du\,dy = \int_0^1 \Big[\frac{y}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big]_{y^2}^{1+y^2} dy$$

$$= \int_0^1 \Big[\frac{y}{3}(1+y^2)^{\frac{3}{2}} - \frac{y}{3}(y^2)^{\frac{3}{2}}\Big]\,dy = \int_0^1 \Big[\frac{y}{3}(1+y^2)^{\frac{3}{2}} - \frac{y^4}{3}\Big]\,dy$$

$$= \int \frac{y}{3}(1+y^2)^{\frac{3}{2}}\,dy - \int_0^1 \frac{y^4}{3}\,dy = \int_1^2 \frac{1}{6}u^{\frac{3}{2}}\,du - \Big[\frac{y^5}{15}\Big]_0^1 = \Big[\frac{1}{15}u^{\frac{5}{2}}\Big]_1^2 - \frac{1}{15}$$

$= \frac{1}{15}\left(2^{\frac{5}{2}} - 1 - 1\right) = \frac{1}{15}\left(4\sqrt{2} - 2\right) = \boxed{\frac{4\sqrt{2}-2}{15}}$

4. $\iint_R \cos(x+2y)\,dA,\ R = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \pi/2\}$

$\iint_R \cos(x+2y)\,dA = \int_0^{\frac{\pi}{2}} \int_0^{\pi} \cos(x+2y)\,dx\,dy = \int_0^{\frac{\pi}{2}} \Big[\sin(x+2y)\Big]_0^{\pi} dy$

$= \int_0^{\frac{\pi}{2}} \Big[\sin(\pi+2y) - \sin(2y)\Big]\,dy = \Big[-\frac{\cos(\pi+2y)}{2} + \frac{\cos(2y)}{2}\Big]_0^{\frac{\pi}{2}}$

$= \Big(-\frac{\cos(2\pi)}{2} + \frac{\cos(\pi)}{2}\Big) - \Big(-\frac{\cos(\pi)}{2} + \frac{1}{2}\Big) = -\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \boxed{-2}$

5. $\iint_R \frac{2y}{x^2+1}\,dA,\ R = \{(x,y) : 0 \le x \le 1, 0 \le y \le \sqrt{x}\}$

$\iint_R \frac{2y}{x^2+1}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \Big[\frac{y^2}{x^2+1}\Big]_0^{\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$

$= \int_1^2 \frac{1}{2}\cdot\frac{1}{u}\,du = \Big[\frac{1}{2}\ln|u|\Big]_1^2 = \frac{1}{2}\Big(\ln 2 - \ln 1\Big) = \boxed{\frac{\ln 2}{2}}$

6. $\iint_R y^3\,dA$, R is the triangular region with vertices $(0,2), (1,1), (3,2)$.

$\iint_R y^3\,dA = \int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 \Big[y^3 x\Big]_{2-y}^{2y-1} dy = \Big[y^3(2y-1) - y^3(2-y)\Big]\,dy$

$= \int_1^2 \Big(3y^4 - 3y^3\Big)\,dx = \Big[\frac{3y^5}{5} - \frac{3y^4}{4}\Big]_1^2 = \frac{96}{5} - 12 - \frac{3}{5} + \frac{3}{4} = \boxed{\frac{147}{20}}$

7. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta$

$\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta = \int_0^{\pi/2} \Big[e^{\sin\theta} r\Big]_0^{\cos\theta} d\theta = \int_0^{\pi/2} e^{\sin\theta}\cos\theta\,d\theta$

$= \Big[e^{\sin\theta}\Big]_0^{\frac{\pi}{2}} = e^{\sin\frac{\pi}{2}} - e^0 = \boxed{e-1}$

8. $\int_0^1 \int_x^{2-x} (x^2 - y)\,dy\,dx$

$$\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx = \int_0^1 \Big[x^2 y - \frac{y^2}{2}\Big]_x^{2-x} dx$$

$$= \int_0^1 \Big[\Big(x^2(2-x) - \frac{(2-x)^2}{2}\Big) - \Big(x^3 - \frac{x^2}{2}\Big)\Big] dx$$

$$= \int_0^1 \Big(-2x^3 + 2x^2 + 2x - 2\Big) dx = \Big[-\frac{x^4}{2} + \frac{2x^3}{3} + x^2 - 2x\Big]_0^1$$

$$= -\tfrac{1}{2} + \tfrac{2}{3} + 1 - 2 = \boxed{-\frac{5}{6}}$$

9. Find the volume under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$.

The volume under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$ is given by the integral

$$\int_0^1 \int_{x^4}^{x} (x + 2y)\, dx\, dy = \int_0^1 \Big[xy + y^2\Big]_{x^4}^{x} dx = \int_0^1 (x^2 + x^2) - (x^5 + x^8)\, dx$$

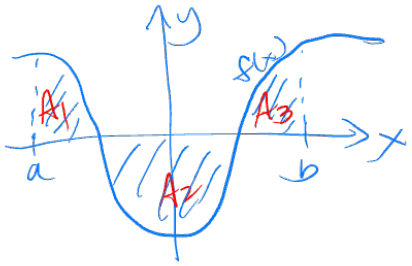
$$= \int_0^1 (2x^2 - x^5 - x^8)\, dx = \Big[\tfrac{2x^3}{3} - \tfrac{x^6}{6} - \tfrac{x^9}{9}\Big]_0^1 = \boxed{\frac{7}{18}}$$

Unit D

Multivariable Integration

D.1 Double Integrals over Rectangles

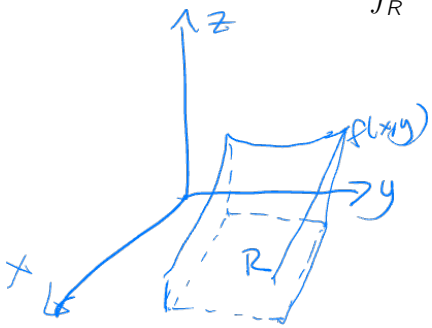
In a function of one variable, the definite integral represents the area under the curve.



$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

What is the 3-dimensional analog to area? Volume

Fact. If $f(x, y)$ is a positive function, then the Volume under the graph of f above region R in the xy-plane is given by $\int_R f\,dx\,dy$. We sometimes write dA instead of dx and dy.



Note. If $f(x, y) = 1$, so that the volume under the graph is always a shape of height 1, then:

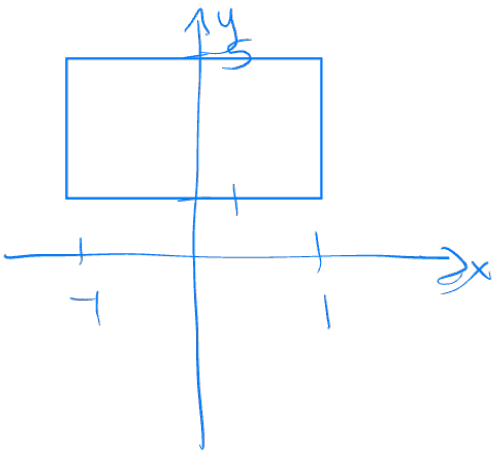
$$\int_R f\,dA = \int_R 1\,dA = \text{ Area of } R.$$

Example. Calculate $\int\int_R 5\,dA$ where $R = \{(x, y) \mid -1 \le x \le 1, 1 \le y \le 5\}$.

$\iint_R 5\,dA = 5 \cdot \iint_R 1\,dA$

$= 5 \cdot (\text{Area of } R)$

$= 5 \cdot 2 \cdot 4 = \boxed{40}$



If R is a rectangle described by $a \le x \le b, c \le y \le d$ and f is continuous on R,

$$\int_R f\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Note. For any function we deal with over a rectangle, the order in which we integrate won't matter. (This is called Fubini's Theorem.)

Note. All the usual properties and rules of integration still apply.

Example. Calculate $\int_1^3 \int_0^1 1 + 2xy\,dx\,dy$

$$\int_1^3 \int_0^1 1+2xy\,dx\,dy = \int_1^3 \left[x + x^2 y\right]_{x=0}^{x=1} dy = \int_1^3 1+y\,dy$$

$$= y + \frac{y^2}{2}\Big]_1^3 = 3 + \frac{9}{2} - \left(1 + \frac{1}{2}\right) = 2 + \frac{8}{2} = \boxed{6}$$

Note: If $f(x,y) = g(x)\cdot h(y)$ and R is a rectangle, then $\int_R f\,dA = \int g(x)\,dx \cdot \int h(y)\,dy$

Example. Calculate $\int\int_R x\sec^2 y\,dA$, $R = \{(x,y) : 0 \le x \le 2, 0 \le y \le \frac{\pi}{4}\}$

$$\iint_R x\sec^2 y\,dA = \int_0^2 x\,dx \cdot \int_0^{\pi/4} \sec^2 y\,dy = \left[\frac{x^2}{2}\right]_0^2 \cdot \left[\tan y\right]_0^{\pi/4}$$

$$= [2-0]\cdot[\tan \pi/4 - \tan 0] = 2\cdot(1-0) = \boxed{2}$$

Example. Calculate $\int\int_R ye^{-xy}\,dA$, $R = [0,2]\times[0,3]$

Note: $ye^{-xy} \ne g(x)\cdot h(y)$

$$\iint_R ye^{-xy}\,dA = \int_0^3 \int_0^2 ye^{-xy}\,dx\,dy = \int_0^3 \int_0^{-2y} e^u\,du\,dy = \int_0^3 \left[e^u\right]_0^{-2y} dy$$

$u = -xy$, $du = -y\,dx$

$$= \int_0^3 -e^{-2y} + e^0\,dy = \frac{e^{-2y}}{2} + y\Big]_0^3 = \frac{e^{-6}}{2} + 3 - \left(\frac{e^0}{2} + 0\right)$$

$u = -2y$ (skipped)

$$= \boxed{\frac{e^{-6}}{2} + \frac{5}{2}}$$

Note: $\int_0^2 \int_0^3 ye^{-xy}\,dy\,dx$ requires integration by parts

Homework D.1 - Double Integrals over Rectangles

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the double integral.

   (a) $\int\int_R \sqrt{2}\, dA,\ R = \{(x,y) : 2 \le x \le 6, -1 \le y \le 5\}$

   $z = \sqrt{2} > 0$ so we can interpret the double integral as the volume of the solid S that lies below the plane $z = \sqrt{2}$ and above the rectangle $[2,6] \times [-1,5]$. Hence, $\int\int_R \sqrt{2}\, dA = \sqrt{2} \cdot 4 \cdot 6 = \boxed{24\sqrt{2}}$

   (b) $\int\int_R (4-2y)\, dA,\ R = [0,1] \times [0,1]$

   $$\int\int_R (4-2y)\, dA = \int_0^1 \int_0^1 (4-2y)\, dx\, dy = \int_0^1 \Big[4x - 2xy\Big]_0^1 dy = \int_0^1 (4-2y)\, dy$$

   $$= \Big[4y - y^2\Big]_0^1 = 4 - 1 = \boxed{3}$$

   (c) $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$

   $$\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 \Big[3x^2y^2 - 2xy\Big]_0^2 dx = \int_1^4 (12x^2 - 4x)\, dx$$

   $$= \Big[4x^3 - 2x^2\Big]_1^4 = (256 - 32) - (4 - 2) = \boxed{222}$$

   (d) $\int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} (\sin x + \sin y)\, dy\, dx$

   $$\int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{2}} (\sin x + \sin y)\, dy\, dx = \int_0^{\frac{\pi}{6}} \Big[y \sin x - \cos y\Big] 0^{\frac{\pi}{2}}\, dx$$

   $$= \int_0^{\frac{\pi}{6}} \Big[\Big(\tfrac{\pi \sin x}{2} - 0\Big) - (0 - 1)\Big]\, dx = \int_0^{\frac{\pi}{6}} \Big(\tfrac{\pi \sin x}{2} + 1\Big)\, dx = \Big[-\tfrac{\pi \cos x}{2} + x\Big]_0^{\frac{\pi}{6}}$$

   $$= -\tfrac{\pi}{2} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{6} + \tfrac{\pi}{2} - 0 = \boxed{\Big(\frac{2}{3} - \frac{\sqrt{3}}{4}\Big)\pi}$$

(e) $\int_{-3}^{3}\int_{0}^{\frac{\pi}{2}}(y+y^2\cos x)\,dx\,dy$

$\int_{-3}^{3}\int_{0}^{\frac{\pi}{2}}(y+y^2\cos x)\,dx\,dy=\int_{-3}^{3}\left[xy+y^2\sin x\right]_{0}^{\frac{\pi}{2}}dy=\int_{-3}^{3}\left(\frac{\pi}{2}y+y^2\right)dy$

$=\left[\frac{\pi}{4}y^2+\frac{1}{3}y^3\right]_{3}^{3}=\left[\left(\frac{9\pi}{4}+9\right)-\left(\frac{9\pi}{4}-9\right)\right]=\boxed{18}$

(f) $\int_{1}^{3}\int_{1}^{5}\frac{\ln y}{xy}\,dy\,dx$

$\int_{1}^{3}\int_{1}^{5}\frac{\ln y}{xy}\,dy\,dx=\int_{1}^{3}\int_{0}^{\ln 5}\frac{u}{x}\,du\,dx=\int_{1}^{3}\left[\frac{u^2}{2x}\right]_{0}^{\ln 5}dx$

$=\int_{1}^{3}\frac{(\ln 5)^2}{2x}\,dx=\left[\frac{(\ln)^5\ln|x|}{2}\right]_{1}^{3}=\boxed{\frac{(\ln)^5\ln 3}{2}}$

Note: Since the integrand can be written as a product of a function of x and a function of y and we are integrating over a rectangle:

$\int_{1}^{3}\int_{1}^{5}\frac{\ln y}{xy}\,dy\,dx=\int_{1}^{3}\frac{1}{x}\,dx\cdot\int_{1}^{5}\frac{\ln y}{y}\,dy=\left[\ln|x|\right]_{1}^{3}\cdot\left[\frac{u^2}{2}\right]_{0}^{\ln 5}=\ln 3\cdot\frac{(\ln 5)^2}{2}$

(g) $\int_{1}^{4}\int_{1}^{2}\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx$

$\int_{1}^{4}\int_{1}^{2}\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx=\int_{1}^{4}\left[x\ln|y|+\frac{y^2}{2x}\right]_{1}^{2}dx=\int_{1}^{4}\left(x\ln 2+\frac{3}{2x}\right)dx=\left[\frac{1}{2}x^2\ln 2+\frac{3}{2}\ln|x|\right]_{1}^{4}$

$=\left(8\ln 2+\frac{3}{2}\ln 4\right)-\left(\frac{1}{2}\ln 2+0\right)=\boxed{\frac{15}{2}\ln 2+\frac{3}{2}\ln 4}$ [this is equal to $\frac{21}{2}\ln 2$]

(h) $\int_{0}^{1}\int_{0}^{1}xy\sqrt{x^2+y^2}\,dy\,dx$

$\int_{0}^{1}\int_{0}^{1}xy\sqrt{x^2+y^2}\,dy\,dx=\int_{0}^{1}x\left[\frac{1}{3}(x^2+y^2)^{\frac{3}{2}}\right]_{0}^{1}dx=\frac{1}{3}\int_{0}^{1}x[(x^2+1)^{\frac{3}{2}}-x^3]\,dx$

$=\frac{1}{3}\int_{0}^{1}[x(x^2+1)^{\frac{3}{2}}-x^4]\,dx=\frac{1}{3}\left[\frac{1}{5}(x^2+1)^{\frac{5}{2}}-\frac{1}{5}x^5\right]_{0}^{1}=\frac{1}{15}\left[(2^{\frac{5}{2}}-1)-(1-0)\right]$

$=\boxed{\frac{2}{15}(2\sqrt{2}-1)}$

(i) $\int\int_R(y+xy^{-2})\,dA$, $R=\{(x,y):0\le x\le 2,1\le y\le 2\}$

$\int\int_R(y+xy^{-2})\,dA=\int_{1}^{2}\int_{0}^{2}(y+xy^{-2})\,dx\,dy=\int_{1}^{2}\left[xy+\frac{1}{2}x^2y^{-2}\right]_{0}^{2}dy$

$=\int_{1}^{2}(2y+2y^{-2})\,dy=\left[y^2-2y^{-1}\right]_{1}^{2}=(4-1)-(1-2)=\boxed{4}$

(j) $\int\int_R \frac{xy^2}{x^2+1}\, dA,\ R = \{(x,y) : 0 \le x \le 1, -3 \le y \le 3\}$

$\int\int_R \frac{xy^2}{x^2+1}\, dA = \int_0^1 \int_3^3 \frac{xy^2}{x^2+1}\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \frac{x}{x^2+1}\, \mathrm{d}x \int_{-3}^3 y^2\, \mathrm{d}y = \Big[\frac{1}{2}\ln(x^2+1)\Big]_0^1 \Big[\frac{1}{3}y^3\Big]_{-3}^3$

$= \frac{1}{2}(\ln 2 - \ln 1) \cdot \frac{1}{3}(27+27) = \boxed{9 \ln 2}$

(k) $\int\int_R x \sin(x+y)\, dA,\ R = \Big[0, \frac{\pi}{6}\Big] \times \Big[0, \frac{\pi}{3}\Big]$

$\int\int_R x \sin(x+y)\, dA = \int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{3}} x \sin(x+y)\, \mathrm{d}y\, \mathrm{d}x = \int_0^{\frac{\pi}{6}} \Big[-x\cos(x+y)\Big]_0^{\frac{\pi}{3}}\, \mathrm{d}x$

$= \int_0^{\frac{\pi}{6}} \Big(-x\cos(x+\frac{\pi}{3}) + x\cos x\Big)\, \mathrm{d}x$

$= x\Big[\sin x - \sin(x+\frac{\pi}{3})\Big]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} [\sin x - \sin(x+\frac{\pi}{3})]\, \mathrm{d}x$ [by integrating by parts separately for each term]

$= \frac{\pi}{6}\Big[\frac{1}{2} - 1\Big] - \Big[-\cos x + \cos(x+\frac{\pi}{3})\Big]_0^{\frac{\pi}{6}}$

$= -\frac{\pi}{12} - \Big[-\frac{\sqrt{3}}{2} + 0 - \Big(-1 + \frac{1}{2}\Big)\Big] = \boxed{\frac{\sqrt{3}-1}{2} - \frac{\pi}{12}}$

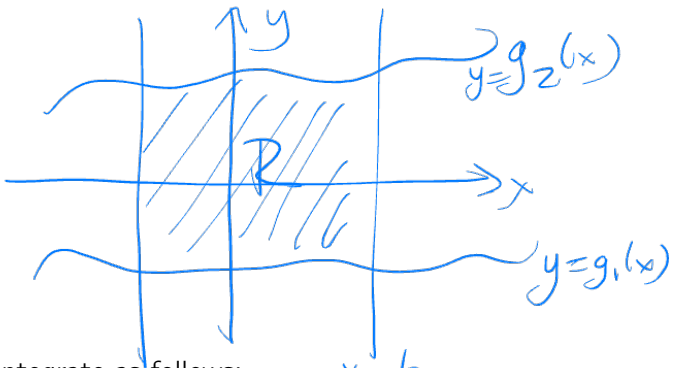
D.2 Double Integrals over General Regions

We wish to be able to integrate over more general regions than rectangles. Very rarely does life fit into nice, neat boxes. We will describe two types of regions and how to integrate over them.

- A plane region R is said to be of **Type I** if it is of the form:

$$R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where g_1, g_2 are continuous functions.



R is a region of Type I

We can integrate as follows:

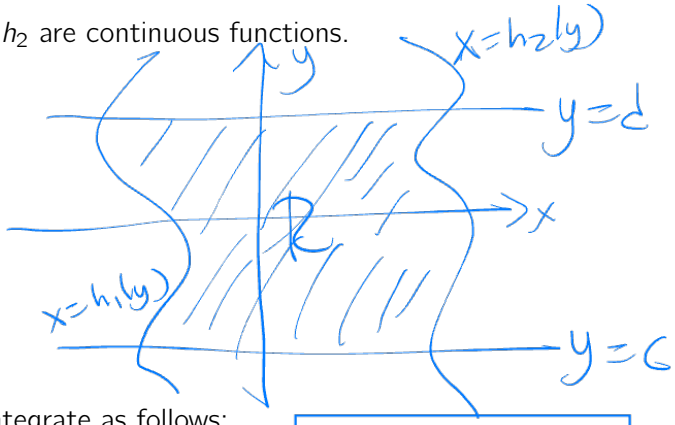
$x=a$ $x=b$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

- A plane region R is said to be of **Type II** if it is of the form:

$$R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

where h_1, h_2 are continuous functions.



R is a region of Type II

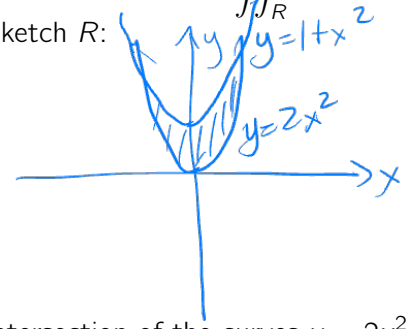
We can integrate as follows:

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Regions can certainly be both types at once, but often one order easier than the other.

Example. Evaluate $\iint_R x + 2y \, dA$ where R is bounded by $y = 2x^2$ and $y = 1 + x^2$.

Sketch R:



R is of both types, but it's easier to think of it as Type I

$R = \{(x,y) \mid -1 \le x \le 1, \; 2x^2 \le y \le 1+x^2\}$

Intersection of the curves $y = 2x^2$ and $y = 1 + x^2$ is given by:

$2x^2 = 1+x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ → $(-1,2)$, $(1,2)$

It is easiest to make this a Type 1 since the functions are of the form "$y =$". We compute:

$$\iint_R (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{-1}^{1}\left[xy+y^2\right]_{2x^2}^{1+x^2} dx$$

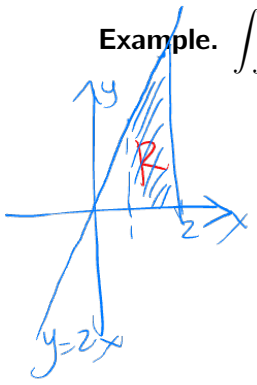
$$= \int_{-1}^{1}\left[x(1+x^2) + (1+x^2)^2 - x\cdot 2x^2 - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4 \, dx = \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \, dx$$

$$= -\frac{3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x \Big]_{-1}^{1} = \frac{-3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1 - \frac{3}{5} + \frac{1}{4} + \frac{2}{3} - \frac{1}{2} + 1$$

$$= \boxed{\frac{32}{15}}$$

Example. $\iint_R \frac{4y}{x^3+2}\,dA$ where $R = \{(x,y) : 1 \le x \le 2, 0 \le y \le 2x\}$



$$\iint_R \frac{4y}{x^3+2}\,dA = \int_1^2\int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2 \frac{2y^2}{x^3+2}\Big]_0^{2x} dx$$

$$= \int_1^2 \frac{2(2x)^2}{x^3+2} - \frac{2\cdot 0^2}{x^3+2}\,dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$$

$u = x^3+2$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$x=1,\ u=3$

$x=2,\ u=8+2=10$

$$= \frac{8}{3}\int_3^{10} \frac{1}{u}\,du = \frac{8}{3}\ln|u|\Big]_3^{10} = \frac{8}{3}(\ln 10 - \ln 3)$$

$$= \boxed{\frac{8}{3}\ln\left(\frac{10}{3}\right)}$$

Example. $\iint_R 2x - y\,dA$ where R is the circle centered at the origin of radius 2.

$R = \{(x,y) \mid x^2+y^2 = 4\}$ is a region of both types. We will rewrite it as Type I

$y^2 = 4-x^2 \Rightarrow y = \pm\sqrt{4-x^2} \Rightarrow R = \{(x,y) \mid -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\}$

$$\iint_R 2x-y\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x-y\,dy\,dx = \int_{-2}^{2}\left[2xy - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 2x\sqrt{4-x^2} - \frac{(4-x^2)}{2} + 2x\sqrt{4-x^2} + \frac{(4-x^2)}{2}\,dx = \int_{-2}^{2} 4x\sqrt{4-x^2}\,dx$$

$u = 4-x^2$, $du = -2x\,dx$

$$= -2\int_0^0 \sqrt{u}\,du = \boxed{0}$$

Example. Sketch the region and switch the order of integration.

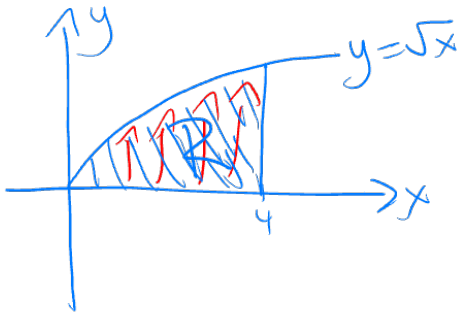
$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$$

Describe R as a Type II region

$0 \le y \le \sqrt{x}$

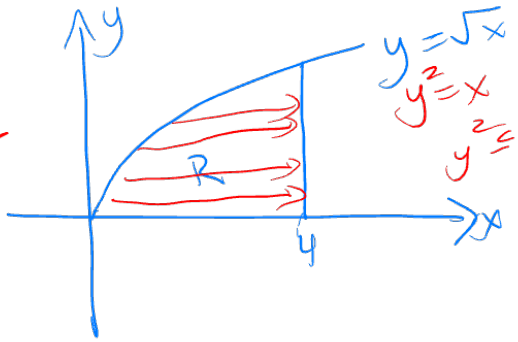
$0 \le x \le 4$

Type I



$R = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le \sqrt{x}\}$

Type II



$R = \{(x,y) \mid 0 \le y \le 2,\ y^2 \le x \le 4\}$

$y^2 \le x \le 4$

$0 \le y \le \sqrt{4}$

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx = \int_0^2 \int_{y^2}^4 f(x,y)\,dx\,dy$$

Worksheet: Multivariable Integration Practice

Homework D.2 - Double Integrals over General Regions

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the double integral.

   (a) $\int_0^2 \int_0^{y^2} x^2 y \, dx \, dy$

   $\int_0^2 \int_0^{y^2} x^2 y \, dx \, dy = \int_0^2 \left[\frac{1}{3}x^3 y\right]_0^{y^2} dy = \int_0^2 \frac{1}{3}y[(y^2)^3 - (0)^3] \, dy = \int_0^2 \frac{1}{3}y^7 \, dy$

   $= \frac{1}{3}\left[\frac{1}{8}y^8\right]_0^2 = \frac{1}{3}(32 - 0) = \boxed{\frac{32}{3}}$

   (b) $\int_0^{\frac{\pi}{2}} \int_0^x x \sin y \, dy \, dx$

   $\int_0^{\frac{\pi}{2}} \int_0^x x \sin y \, dy \, dx = \int_0^{\frac{\pi}{2}} \Big[- x \cos y\Big]_0^x dx = \int_0^{\frac{\pi}{2}} [-x \cos x + x] \, dx$

   $= \int_0^{\frac{\pi}{2}} -x \cos x \, dx + \left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{2}}$

   $= \Big[- (x \sin x + \cos x)\Big]_0^{\frac{\pi}{2}} + \frac{\pi^2}{8}$ (by integrating by parts)

   $= \frac{\pi^2}{8} + \Big(- \frac{\pi}{2} - 0\Big) + (0 + 1) = \boxed{\frac{\pi^2}{8} - \frac{\pi}{2} + 1}$

   (c) $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v} \, dw \, dv$

   $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v} \, dw \, dv = \int_0^1 \left[w\sqrt{1 + e^v}\right]_0^{e^v} dv = \int_0^1 e^v \sqrt{1 + e^v} \, dv = \left[\frac{2}{3}(1 + e^v)^{\frac{3}{2}}\right]_0^1$

   $= \frac{2}{3}(1 + e)^{\frac{3}{2}} - \frac{2}{3}(1 + 1)^{\frac{3}{2}} = \boxed{\frac{2}{3}(1 + e)^{\frac{3}{2}} - \frac{4}{3}\sqrt{2}}$

(d) $\int\int_R \frac{y}{x^2+1}\,dA$, $R = \{(x,y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$

$$\int\int_R \frac{y}{x^2+1}\,dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \left[\frac{1}{x^2+1}\cdot\frac{y^2}{2}\right]_0^{\sqrt{x}} = \frac{1}{2}\int_0^4 \frac{x}{x^2+1}\,dx$$

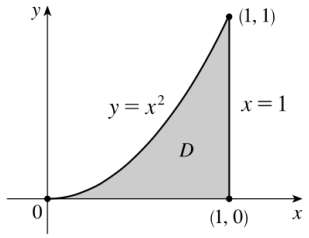
$$= \frac{1}{2}\left[\frac{1}{2}\ln|x^2+1|\right]_0^4 = \frac{1}{4}\left[\ln(x^2+1)\right]_0^4 = \frac{1}{4}(\ln 17 - \ln 1) = \boxed{\frac{1}{4}\ln 17}$$

(e) $\int\int_R e^{-y^2}\,dA$, $R = \{(x,y) : 0 \le y \le 3, 0 \le x \le y\}$

$$\int\int_R e^{-y^2}\,dA = \int_0^3 \int_0^y e^{-y^2}\,dx\,dy = \int_0^3 \left[xe^{-y^2}\right]_0^y dy = \int_0^3 \left(ye^{-y^2} - 0\right) dy = \int_0^3 ye^{-y^2}\,dy$$

$$= \left[-\frac{1}{2}e^{-y^2}\right]_0^3 = -\frac{1}{2}(e^{-9} - e^0) = \boxed{\frac{1}{2}(1 - e^{-9})}$$

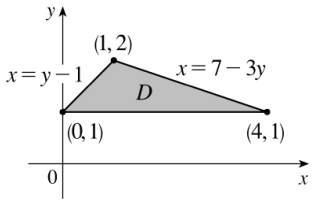
(f) $\int\int_D x\cos y\,dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$



$$\int\int_D x\cos y\,dA = \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \left[x\sin y\right]_0^{x^2} dx = \int_0^1 x\sin x^2\,dx$$

$$= \left[-\frac{1}{2}\cos x^2\right]_0^1 = -\frac{1}{2}(\cos 1 - \cos 0) = \boxed{\frac{1}{2}(1 - \cos 1)}$$

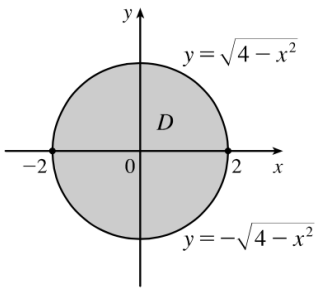
(g) $\int\int_D y^2\,dA$, D is the triangular region with vertices $(0,1)$, $(1,2)$, $(4,1)$.



$$\int\int_D y^2\,dA = \int_1^2 \int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2 \left[xy^2\right]_{y-1}^{7-3y} dy = \int_1^2 \left[(7-3y) - (y-1)\right]y^2\,dy$$

$$= \int_1^2 (8y^2 - 4y^3)\,dy = \left[\frac{8}{3}y^3 - y^4\right]_1^2 = \frac{64}{3} - 16 - \frac{8}{3} + 1 = \boxed{\frac{11}{3}}$$

(h) $\int\int_D (2x-y)\,dA$, D is bounded by the circle with center the origin and radius 2.



$$\int\int_D (2x-y)\,dA = \int_2^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx = \int_{-2}^2 \left[2xy - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

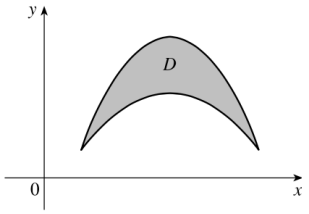
$$= \int_{-2}^2 \left[2x\sqrt{4-x^2} - \tfrac{1}{2}(4-x^2) + 2x\sqrt{4-x^2} + \tfrac{1}{2}(4-x^2)\right] dx = \int_{-2}^2 4x\sqrt{4-x^2}\,dx$$

$$= \left[-\tfrac{4}{3}(4-x^2)^{\frac{3}{2}}\right]_{-2}^2 = \boxed{0}$$

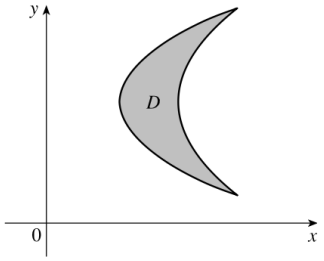
Alternatively, note that $4x\sqrt{4-x^2}$ is an odd function so $\int_{-2}^2 4x\sqrt{4-x^2}\,dx = 0$

2. Draw an example of a region that is

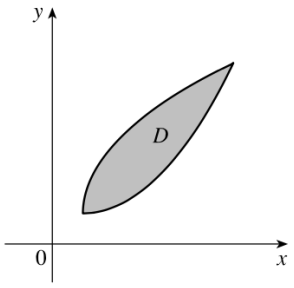
(a) type I but not type II



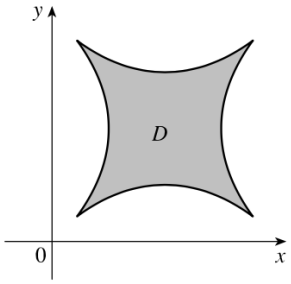
(b) type II but not type I



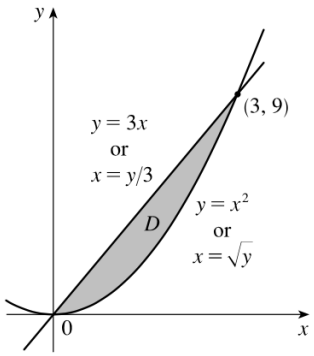
(c) both type I and type II



(d) neither type I nor type II



3. Let D be the region enclosed by the curves $y = x^2$ and $y = 3x$. Express D as a region of type I and also a region of type II. Then evaluate the double integral $\int\int_D xy\,\mathrm{d}A$ in two ways.



The curves $y = x^2$ and $y = 3x$ intersect at points $(0, 0)$ and $(3, 9)$. As a type I region, D is enclosed by the lower boundary $y = x^2$ and the upper boundary $y = 3x$ for

$0 \le x \le 3$, so $D = \{(x,y) : 0 \le x \le 3, x^2 \le y \le 3x\}$. If we describe D as a type II region, D is enclosed by the left boundary $x = \frac{y}{3}$ and the right boundary $x = \sqrt{y}$ for $0 \le y \le 9$, so $D = \{(x,y) : 0 \le y \le 9, \frac{y}{3} \le x \le \sqrt{y}\}$. Thus

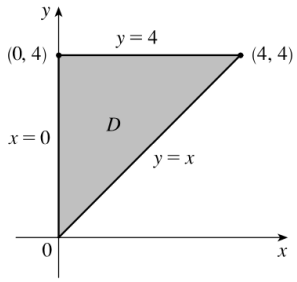
$$\int\int_D xy \, dA = \int_0^3 \int_{x^2}^{3x} xy \, dy \, dx = \int_0^3 \left[x \cdot \frac{y^2}{2}\right]_{x^2}^{3x} dx = \frac{1}{2}\int_0^3 x(9x^2 - x^4) \, dx$$

$$= \frac{1}{2}\int_0^3 (9x^3 - x^5) \, dx = \frac{1}{2}\left[9 \cdot \frac{x^4}{4} - \frac{x^6}{6}\right]_0^3 = \frac{1}{2}\left[\left(\frac{9}{4} \cdot 81 - \frac{1}{6} \cdot 729\right) - 0\right] = \boxed{\frac{243}{8}}$$

$$\int\int_D xy \, dA = \int_0^9 \int_{\frac{y}{3}}^{\sqrt{y}} xy \, dx \, dy = \int_0^9 \left[\frac{1}{2}x^2 y\right]_{\frac{y}{3}}^{\sqrt{y}} dy = \frac{1}{2}\int_0^9 \left(y - \frac{1}{9}y^2\right) y \, dy$$

$$= \frac{1}{2}\int_0^9 \left(y^2 - \frac{1}{9}y^2\right) dy = \frac{1}{2}\left[\frac{1}{3}y^3 - \frac{1}{9} \cdot \frac{y^4}{4}\right]_0^9 = \frac{1}{2}\left[\left(\frac{1}{3} \cdot 729 - \frac{1}{36} \cdot 6561\right) - 0\right] = \boxed{\frac{243}{8}}$$

4. Let D be the region bounded by $y = x$, $y = 4$, and $x = 0$. Set up integrals for both orders of integration. Then evaluate the double integral $\int\int_D y^2 e^{xy} \, dA$ using the easier order and explain why it's easier.



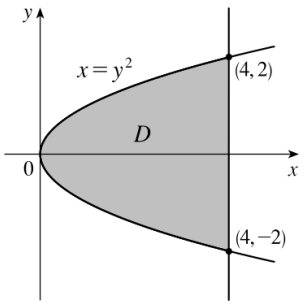
As a type I region, $D = \{(x,y) : 0 \le x \le 4, x \le y \le 4\}$ and $\int\int_D y^2 e^{xy} \, dA = \int_0^4 \int_x^4 y^2 e^{xy} \, dy \, dx$. As a type II region, $D = \{(x,y) : 0 \le y \le 4, 0 \le x \le y\}$ and $\int\int_D y^2 e^{xy} \, dA = \int_0^4 \int_0^y y^2 e^{xy} \, dx \, dy$.

Evaluating $\int y^2 e^{xy} \, dy$ requires integration by parts whereas $\int y^2 e^{xy} \, dx$ does not, so the integral corresponding to D as a type II region appears easier to evaluate.

$$\int\int_D y^2 e^{xy} \, dA = \int_0^4 \int_0^y y^2 e^{xy} \, dx y = \int_0^4 \left[y e^{xy}\right]_0^y dy = \int_0^4 \left(y e^{y^2} - y\right) dy$$

$$= \left[\frac{1}{2}e^{y^2} - \frac{y^2}{2}0^4 = \left(\frac{1}{2}e^{16} - 8\right) - \left(\frac{1}{2} - 0\right) = \boxed{\frac{1}{2}e^{16} - \frac{17}{2}}\right.$$

5. Find the volume of the solid under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.

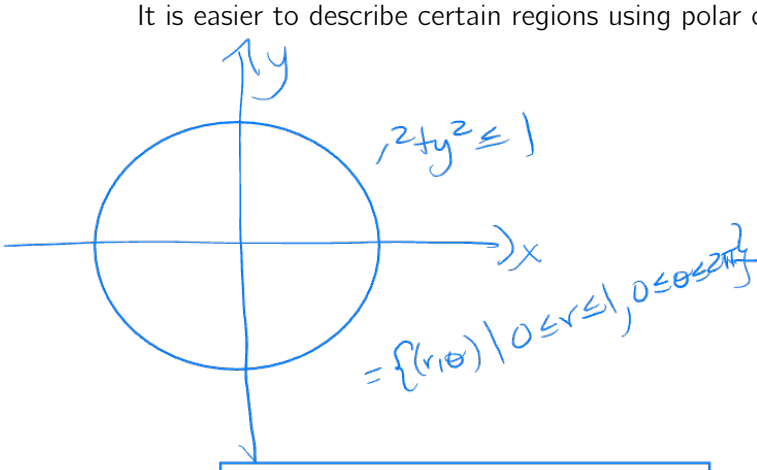


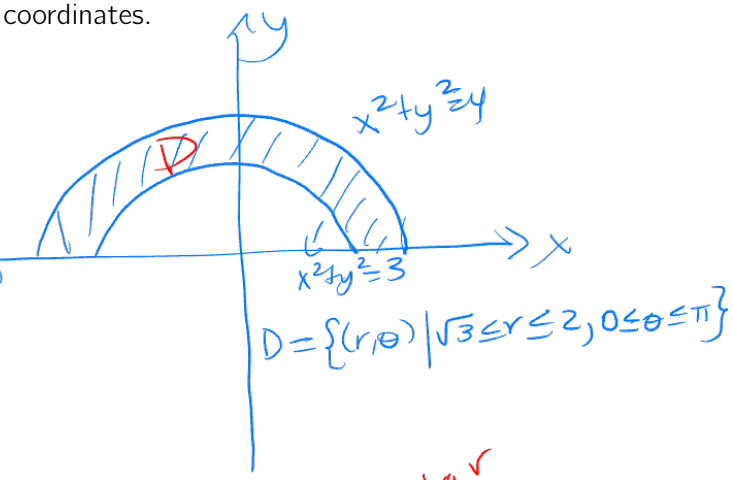
$$V = \int_{-2}^{2}\int_{y^2}^{4}(1 + x^2y^2)\,dx\,dy = \int_{-2}^{2}\left[x + \tfrac{1}{3}x^3y^2\right]_{y^2}^{4} dy = \int_{-2}^{2}\left(4 + \tfrac{61}{3}y^2 - \tfrac{1}{3}y^8\right) dy$$

$$= \left[4y + \tfrac{61}{9}y^3 - \tfrac{1}{27}y^9\right]_{-2}^{2} = 8 + \tfrac{488}{9} - \tfrac{512}{27} + 8 + \tfrac{488}{9} - \tfrac{512}{27} = \boxed{\frac{2336}{27}}$$

D.3 Double Integrals in Polar Coordinates

It is easier to describe certain regions using polar coordinates.





Recall. $r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$

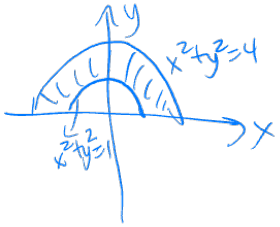
Definition. A **polar rectangle** is a region described by $R = \{(r,\theta) : a \le r \le b, \alpha \le \theta \le \beta\}$.

Fact. If R is a polar rectangle, $\displaystyle\int\int_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$

extra r

*order matters

Example. Evaluate $\displaystyle\int\int_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$R=\{(r,\theta)\mid 1\le r\le 2,\ 0\le\theta\le\pi\}$

$$\iint_R (3x+4y^2)\,dA = \int_0^\pi\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

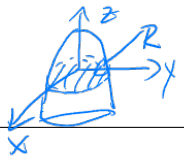
$$= \int_0^\pi\int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\,dr\,d\theta = \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta\right]_1^2 d\theta$$

$$= \int_0^\pi 8\cos\theta + 16\sin^2\theta - \cos\theta - \sin^2\theta\,d\theta = \int_0^\pi 7\cos\theta + 15\sin^2\theta\,d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\cdot\frac{1}{2}(1-\cos 2\theta)\,d\theta = \int_0^\pi 7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta\,d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\theta - \frac{15}{2}\cdot\frac{\sin 2\theta}{2}\right]_0^\pi = 7\sin\pi + \frac{15}{2}\pi - \frac{15}{2}\cdot\frac{\sin 2\pi}{2} - \left(7\sin 0 + \frac{15}{2}\cdot 0 - \frac{15}{2}\cdot\frac{\sin 0}{2}\right)$$

$$= 0 + \frac{15}{2}\pi - 0 - (0+0-0) = \boxed{\frac{15\pi}{2}}$$



$r^2 = x^2 + y^2$

Example. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

If $z=0$, $x^2+y^2=1$ $\Rightarrow$ The paraboloid intersects the circle $x^2+y^2=1$ so the solid lies under the paraboloid and above the disk $x^2+y^2 \le 1$.

$R = \{(x,y) \mid x^2+y^2 \le 1\} = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

$$V = \iint_R 1 - x^2 - y^2 \, dA = \int_0^{2\pi}\int_0^1 (1-r^2) r \, dr\, d\theta = \int_0^{2\pi}\int_0^1 r - r^3 \, dr\, d\theta = \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{1}{4} - (0-0)\, d\theta = \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{1}{4}\theta\Big]_0^{2\pi} = \frac{2\pi}{4} - \frac{0}{4} = \boxed{\pi/2}$$

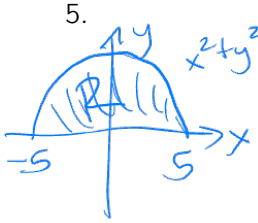
Note: Polar rectangles are analogous to rectangles

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r - r^3 \, dr$$

$$= \left[\theta\right]_0^{2\pi} \cdot \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

$$= 2\pi \cdot \frac{1}{4} = \pi/2$$

Example. Evaluate $\int\int_R x^2 y \, dA$, where R is the top half of the disk with center the origin and radius 5.



$R = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$

$$\iint_R x^2 y \, dA = \int_0^{\pi}\int_0^5 r^2\cos^2\theta \, r\sin\theta \, r \, dr\, d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \cdot \int_0^5 r^4 \, dr = \int_1^{-1} -u^2 \, du \cdot \left[\frac{r^5}{5}\right]_0^5$$

$u = \cos\theta$
$du = -\sin\theta \, d\theta$
$\theta = 0 \Rightarrow u = \cos 0 = 1$
$\theta = \pi \Rightarrow u = \cos\pi = -1$

$$= \left[\frac{u^3}{3}\right]_{-1}^{1} \cdot \left[\frac{5^5}{5} - \frac{0^5}{5}\right]$$

$$= \left[\frac{1}{3} + \frac{1}{3}\right] \cdot 5^4$$

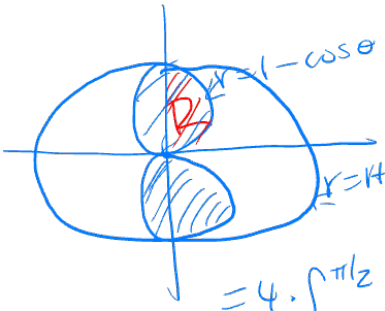
$$= \frac{2}{3} \cdot 625 = \boxed{\frac{1250}{3}}$$

Area of $R = \iint_R 1\,dA$

Example. Use a double integral to find the area of the region enclosed by both of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

$f(x,y) = 1$

Not a polar rectangle



$R = \{(r,\theta) \mid 0 \le r \le 1-\cos\theta,\ 0 \le \theta \le \pi/2\}$

$$A = 4 \cdot \iint_R 1\,dA = 4 \cdot \int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 4 \cdot \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta = 4 \cdot \int_0^{\pi/2} \frac{(1-\cos\theta)^2}{2} - \frac{0^2}{2}\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta\,d\theta = 2 \cdot \int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta$$

$$= 2 \cdot \left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 \cdot \left[\frac{3}{2}\cdot\frac{\pi}{2} - 2\sin\frac{\pi}{2} + \frac{\sin\pi}{4} - (0 - 0 + 0)\right]$$

$$= \boxed{\frac{3\pi}{2} - 4}$$

Example. Use polar coordinates to find the volume of the solid below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \le 1$.

$z = 4 - 2x - y = f(x,y)$

$$V = \iint_{x^2+y^2\le 1} z\,dA = \iint_{x^2+y^2\le 1} 4 - 2x - y\,dA$$

$R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$$V = \int_0^{2\pi}\int_0^1 (4 - 2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 4r - 2r^2\cos\theta - r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} 2r^2 - \frac{2r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\,\Big]_0^1\,d\theta = \int_0^{2\pi} 2 - \frac{2}{3}\cos\theta - \frac{1}{3}\sin\theta\,d\theta$$

$$= 2\theta - \frac{2}{3}\sin\theta + \frac{1}{3}\cos\theta\,\Big]_0^{2\pi} = 4\pi - 0 + \frac{1}{3} - 0 + 0 - \frac{1}{3}$$

$$= \boxed{4\pi}$$

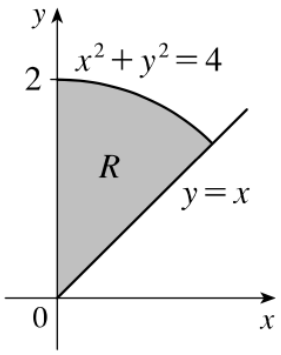
Homework D.3 - Double Integrals in Polar Coordinates

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the given integral by changing to polar coordinates.

   (a) $\int\int_R (2x - y)\,dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.



   The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \left\{(r,\theta) : 0 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\right\}$. Thus,

   $$\int\int_R (2x-y)\,dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 (2r\cos\theta - r\sin\theta) r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 (2r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

   $$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[2\cdot\frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta\right]_0^2 d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{16}{3}\cos\theta - \frac{8}{3}\sin\theta\right) = \left[\frac{16}{3}\sin\theta + \frac{8}{3}\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

   $$= \frac{16}{3} + 0 - \frac{16}{3}\cdot\frac{\sqrt{2}}{2} - \frac{8}{3}\cdot\frac{\sqrt{2}}{2} = \boxed{\frac{16}{3} - 4\sqrt{2}}$$

   (b) $\int\int_R \sin(x^2 + y^2)\,dA$, where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3.

$$\int\int_R \sin(x^2+y^2)\,\mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_1^3 \sin(r^2) r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left[-\tfrac{1}{2}\cos(r^2)\right]_1^3 \mathrm{d}\theta \text{ [}u\text{-substitution]}$$

$$= \int_0^{\frac{\pi}{2}} \left(-\tfrac{1}{2}\cos(9) + \tfrac{1}{2}\cos(1)\right)\mathrm{d}\theta = \left[-\tfrac{\theta}{2}\cos(9) + \tfrac{\theta}{2}\cos(1)\right]_0^{\frac{\pi}{2}} = \tfrac{-\pi\cos(9)}{4} + \tfrac{\pi\cos(1)}{4}$$

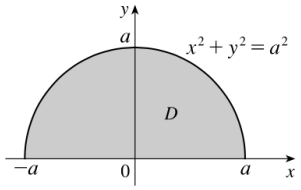
$$= \boxed{\frac{\pi}{4}(\cos 1 - \cos 9)}$$

(c) $\int\int_D e^{-x^2-y^2}\,\mathrm{d}A$, where D is the region bounded by the semi-circle $x = \sqrt{4-y^2}$ and the y-axis.

$$\int\int_D e^{-x^2-y^2}\,\mathrm{d}A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 \mathrm{d}\theta \text{ [}u\text{-substitution]}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-\tfrac{1}{2}e^{-4} + \tfrac{1}{2}\right)\mathrm{d}\theta = \left[\left(-\tfrac{1}{2}e^{-4} + \tfrac{1}{2}\right)\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{\frac{\pi}{2}(1-e^{-4})}$$

(d) $\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,\mathrm{d}x\,\mathrm{d}y$

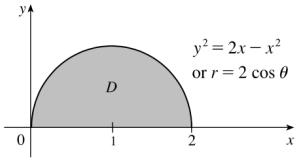


$$\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi} \int_0^a (2r\cos\theta + r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{\pi} \int_0^a (2r^2\cos\theta + r^2\sin\theta)\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\pi} \left[\tfrac{r^3}{3}\left(2\cos\theta - \sin\theta\right)\right]_0^a \mathrm{d}\theta$$

$$= \int_0^{\pi} \left[\tfrac{a^3}{3}\left(2\cos\theta + \sin\theta\right)\right]\mathrm{d}\theta = \left[\tfrac{a^3}{3}\left(2\sin\theta - \cos\theta\right)\right]_0^{\pi} = \tfrac{a^3}{3}\left(0+1-0+1\right) = \boxed{\frac{2a^3}{3}}$$

(e) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$

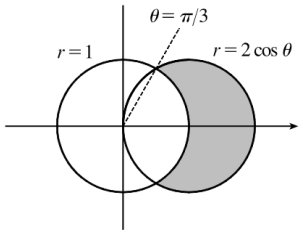


$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\tfrac{1}{3}r^3\right]_0^{2\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \left(\tfrac{8}{3}\cos^3\theta\right) d\theta$$

$$\tfrac{8}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\,d\theta = \tfrac{8}{3}\Big[\sin\theta - \tfrac{1}{3}\sin^3\theta\Big]_0^{\frac{\pi}{2}} = \boxed{\frac{16}{9}}$$

2. Use a double integral to find the area of the region.

(a) The region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.



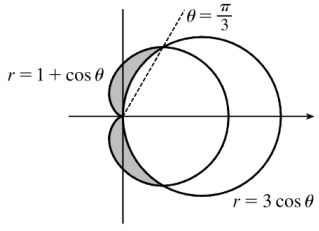
The circle $(x-1)^2+y^2=1$ can be rewritten as $x^2+y^2=2x$. In polar coordinates, $r^2=2r\cos\theta$, which implies $r=2\cos\theta$ (since r is clearly not 0). The circle $x^2+y^2=1$ is $r=1$ in polar coordinates. The curves intersect in the first quadrant when $2\cos\theta=1$, that is, $\theta=\frac{\pi}{3}$, so the portion of the region in the first quadrant is given by $D=\left\{(r,\theta) : 1\le r\le 2\cos\theta, 0\le\theta\le\frac{\pi}{3}\right\}$. By symmetry, the total area is twice the area of D:

$$2\int\int_D dA = 2\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{3}} \left[\tfrac{1}{2}r^2\right]_1^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta = \int_0^{\frac{\pi}{3}} \left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 1\right] d\theta = \int_0^{\frac{\pi}{3}} (1+2\cos 2\theta)\,d\theta$$

$$= \Big[\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}} = \boxed{\frac{\pi}{3}+\frac{\sqrt{3}}{2}}$$

(b) The region inside the cardioid $r=1+\cos\theta$ and outside the circle $r=3\cos\theta$.



The region lies between the polar curves in quadrants I and IV, but in quadrants II and III the region is enclosed by the cardioid. In the first quadrant, $1+\cos\theta = 3\cos\theta$ when $\cos\theta = \frac{1}{2}$, that is, $\theta = \frac{\pi}{3}$, so the area of the region inside the cardioid and outside the circle is

$$A_1 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\tfrac{1}{2}r^2\right]_{3\cos\theta}^{1+\cos\theta} d\theta = \tfrac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + 2\cos\theta - 8\cos^2\theta)\,d\theta$$

$$= \tfrac{1}{2}\Big[\theta + 2\sin\theta - 8\Big(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\Big)\Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \Big[-\tfrac{3}{2}\theta + \sin\theta - \sin 2\theta\Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \Big(-\tfrac{3\pi}{4} + 1 - 0\Big) - \Big(-\tfrac{\pi}{2} + \tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}\Big) = 1 - \tfrac{\pi}{4}$$

The area of the region in the second quadrant is

$$A_2 = \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{1+\cos\theta} r\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\pi} \left[\tfrac{1}{2}r^2\right]_{0}^{1+\cos\theta} d\theta = \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \tfrac{1}{2}\Big[\theta + 2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\Big]_{\frac{\pi}{2}}^{\pi} = \tfrac{1}{2}\Big(\tfrac{3\pi}{4} - 2\Big) = \tfrac{3\pi}{8} - 1$$

By symmetry, the total area is $A = 2(A_1 + A_2) = 2\Big(1 - \tfrac{\pi}{4} + \tfrac{3\pi}{8} - 1\Big) = \boxed{\dfrac{\pi}{4}}$

3. Use polar coordinates to find the volume of the given solid.

   (a) Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \le 25$.

   $$V = \iint_{x^2+y^2\le 25} (x^2 + y^2)\,dA = \int_0^{2\pi} \int_0^5 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \left[\tfrac{r^4}{4}\right]_0^5 d\theta$$

   $$= \int_0^{2\pi} \left(\tfrac{625}{4}\right) d\theta = \left[\tfrac{625}{4}\theta\right]_0^{2\pi} = \boxed{\frac{625}{2}\pi}$$

   (b) Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \le x^2 + y^2 \le 4$.

   $$V = \iint_{1\le x^2+y^2\le 4} \sqrt{x^2 + y^2}\,dA = \int_0^{2\pi} \int_1^2 r^2\,dr\,d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}r^3\right]_1^2 d\theta$$

   $$= \int_0^{2\pi} \left(\tfrac{8}{3} - \tfrac{1}{3}\right) d\theta = \left[\tfrac{7}{3}\theta\right]_0^{2\pi} = \boxed{\frac{14}{3}\pi}$$

D.4 Triple Integrals

We can define triple integrals for functions of three variables. When we defined double integrals, we started integrating over a rectangular region. What would be the analog in three variables?

A rectangular box

Theorem. If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Note. For any function we deal with over a rectangular box, the order in which we integrate won't matter. (This is called Fubini's Theorem for Triple Integrals.)

Note. All the usual properties and rules of integration still apply.

Example. Evaluate $\iiint_B (xyz^2)\, dV$, where $B = \{(x, y, z) : 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$.

$$\iiint_B (xyz^2)\, dV = \int_0^3 \int_0^1 \int_0^2 xyz^2\, dx\, dy\, dz = \int_0^2 x\, dx \cdot \int_0^1 y\, dy \cdot \int_0^3 z^2\, dz$$

$$= \left[\frac{x^2}{2}\right]_0^2 \cdot \left[\frac{y^2}{2}\right]_0^1 \cdot \left[\frac{z^3}{3}\right]_0^3 = 2 \cdot \frac{1}{2} \cdot 3^2 = \boxed{9}$$

- A solid region E is said to be of **Type I** if it is of the form:

$$E = \{(x, y, z) : a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}$$

or

$$E = \{(x, y, z) : c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x, y) \le z \le u_2(x, y)\}$$

where $g_1, g_2, h_1, h_2, u_1, u_2$ are continuous functions.

Projecting solid E onto xy-plane

In the first case, we can integrate as follows:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

In the second case, we can integrate as follows:

$$\int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

- A solid region E is said to be of **Type II** if it is of the form:

$$E = \{(x,y,z) : a \le y \le b, g_1(y) \le z \le g_2(y), u_1(y,z) \le x \le u_2(y,z)\}$$

or

$$E = \{(x,y,z) : c \le z \le d, h_1(z) \le y \le h_2(z), u_1(y,z) \le x \le u_2(y,z)\}$$

where $g_1, g_2, h_1, h_2, u_1, u_2$ are continuous functions.

Projecting E onto yz-plane

In the first case, we can integrate as follows:

$$\int_a^b \int_{g_1(y)}^{g_2(y)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dz\,dy$$

In the second case, we can integrate as follows:

$$\int_c^d \int_{h_1(z)}^{h_2(z)} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

- A solid region E is said to be of **Type III** if it is of the form:

$$E = \{(x,y,z) : a \le x \le b, g_1(x) \le z \le g_2(x), u_1(x,z) \le y \le u_2(x,z)\}$$

or

$$E = \{(x,y,z) : c \le z \le d, h_1(z) \le x \le h_2(z), u_1(x,z) \le y \le u_2(x,z)\}$$

where $g_1, g_2, h_1, h_2, u_1, u_2$ are continuous functions.

Projecting E onto xz-plane

In the first case, we can integrate as follows:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dz\,dx$$

In the second case, we can integrate as follows:

$$\int_c^d \int_{h_1(z)}^{h_2(z)} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dx\,dz$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Example. Evaluate the integral $\iiint_E \frac{z}{x^2+z^2}\,dV$, where E={(x,y,z): $1 \le y \le 4, y \le z \le 4, 0 \le x \le z$}

Type II

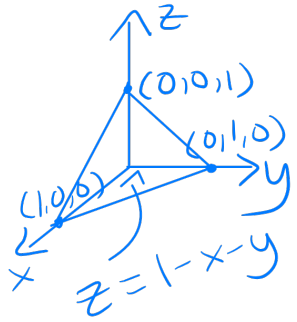
$$\iiint_E \frac{z}{x^2+z^2}\,dV = \int_1^4\int_y^4\int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_1^4\int_y^4 z\cdot\frac{1}{z}\cdot\tan^{-1}\left(\frac{x}{z}\right)\Big]_0^z\,dz\,dy$$

$$= \int_1^4\int_y^4 \left[\tan^{-1}(1)-\tan^{-1}0\right]dz\,dy = \int_1^4\int_y^4 \frac{\pi}{4}\,dz\,dy = \int_1^4 \frac{\pi}{4}z\Big]_y^4\,dy$$

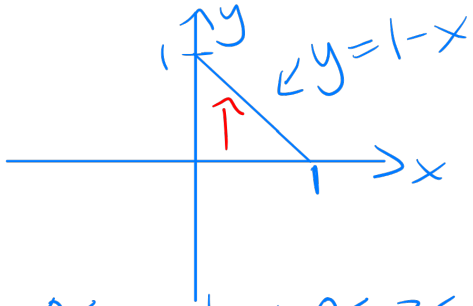
$$= \int_1^4 \pi - \frac{\pi y}{4}\,dy = \pi y - \frac{\pi y^2}{8}\Big]_1^4 = 4\pi - \frac{16\pi}{8} - \left(\pi - \frac{\pi}{8}\right)$$

$$= 3\pi - \frac{15\pi}{8} = \boxed{\frac{9\pi}{8}}$$

Example. Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.



projecting onto xy-plane →



Type I

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$$

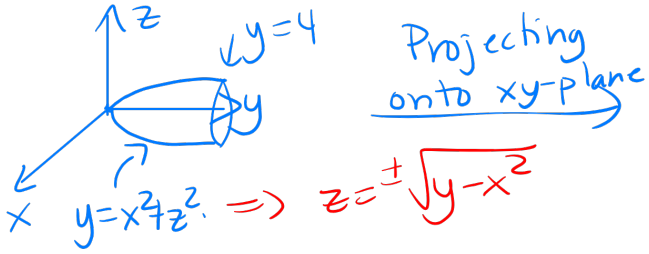
$$\iiint_E z\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\frac{z^2}{2}\Big]_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\frac{(1-x-y)^2}{2}\,dy\,dx \overset{u=1-x-y}{=} \int_0^1 -\frac{(1-x-y)^3}{2\cdot 3}\Big]_0^{y=1-x}\,dx$$

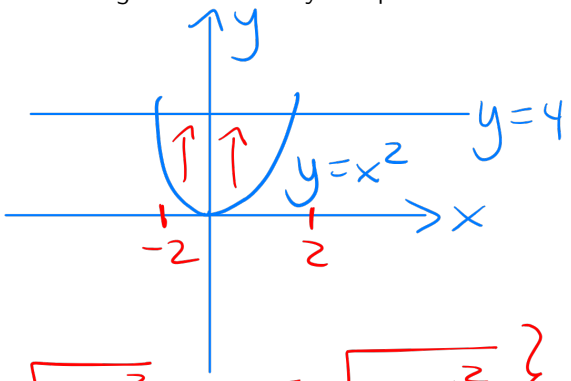
$$= \int_0^1 -\frac{(1-x-1+x)^3}{2\cdot 3} + \frac{(1-x)^3}{6}\,dx = \int_0^1 \frac{(1-x)^3}{6}\,dx \overset{u=1-x}{=} -\frac{(1-x)^4}{24}\Big]_0^1$$

$$= -0 + \frac{1}{24} = \boxed{\frac{1}{24}}$$

Example. Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.



Projecting onto xy-plane

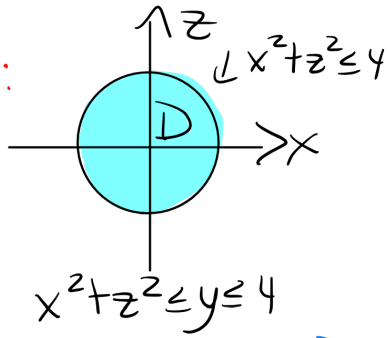


$E = \{(x,y,z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y-x^2} \le z \le \sqrt{y-x^2}\}$ Type I

$$\iiint_E \sqrt{x^2+z^2}\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx$$

← difficult to find

Project onto another plane! Projecting onto xz-plane: (Type III)



$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right) dA$$

$$= \iint_D \sqrt{x^2+z^2}\cdot y \Big]_{x^2+z^2}^{4} dA = \iint_D 4\sqrt{x^2+z^2} - (x^2+z^2)^{3/2}\,dA$$

$x = r\cos\theta$
$z = r\sin\theta$
$x^2+z^2 = r^2$
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

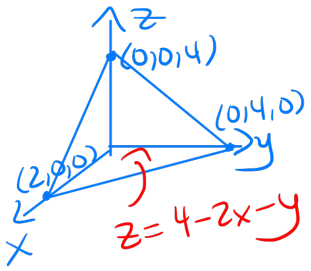
$$= \int_0^{2\pi}\int_0^2 \left[4\sqrt{r^2} - (r^2)^{3/2}\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4r^2 - r^4)\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 4r^2 - r^4\,dr = \theta\Big]_0^{2\pi} \cdot \left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2$$

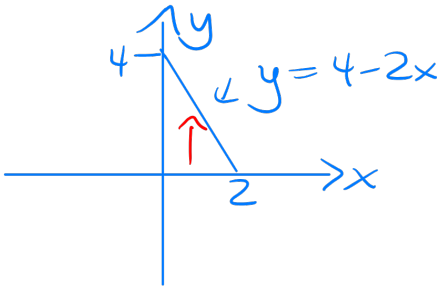
$$= 2\pi \cdot \left[\frac{32}{3} - \frac{32}{5}\right] = 64\pi\left(\frac{2}{15}\right) = \boxed{\frac{128\pi}{15}}$$

$\Rightarrow f(x,y,z)=1$ $\quad x=0,\ y=0,\ z=0$

Example. Use a triple integral to find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.



Projecting onto xy-plane



$$E=\{(x,y,z)\mid 0\le x\le 2,\ 0\le y\le 4-2x,\ 0\le z\le 4-2x-y\} \quad \text{(Type I)}$$

$$V=\iiint_E 1\,dV=\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} 1\,dz\,dy\,dx$$

$$=\int_0^2\int_0^{4-2x} z\Big]_0^{4-2x-y}dy\,dx=\int_0^2\int_0^{4-2x} 4-2x-y\,dy\,dx$$

$$=\int_0^2 4y-2xy-\frac{y^2}{2}\Big]_0^{4-2x}dx=\int_0^2 4(4-2x)-2x(4-2x)-\frac{(4-2x)^2}{2}\,dx$$

$$=\int_0^2 16-8x-8x+4x^2-\frac{1}{2}\cdot(16-16x+4x^2)\,dx$$

$$=\int_0^2 8-8x+2x^2\,dx=8x-4x^2+\frac{2x^3}{3}\Big]_0^2$$

$$=16-16+\frac{16}{3}-(0-0+0)=\boxed{\frac{16}{3}}$$

Homework D.4 - Triple Integrals

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the triple integral:

   (a) $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \, dz \, dx \, dy$

   $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \, dz \, dx \, dy = \int_0^1 \int_y^{2y} \Big[6xyz\Big]_0^{x+y} dx \, dy = \int_0^1 \int_y^{2y} 6xy(x+y) \, dx \, dy$

   $= \int_0^1 \int_y^{2y} (6x^2y + 6xy^2) \, dx \, dy = \int_0^1 \Big[2x^3y + 3x^2y\Big]_y^{2y} = \int_0^1 23y^3 \, dy$

   $= \Big[\frac{23}{5}y^5\Big]_0^1 = \boxed{\frac{23}{5}}$

   (b) $\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1} \, dx \, dz \, dy$

   $\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1} \, dx \, dz \, dy = \int_0^1 \int_0 1 \Big[\frac{z}{y+1} \cdot x\Big]_0^{\sqrt{1-z^2}} dz \, dy = \int_0^1 \int_0^1 \frac{z\sqrt{1-z^2}}{y+1} \, dz \, dy$

   $= \int_0^1 \Big[\frac{-\frac{1}{3}(1-z^2)^{\frac{3}{2}}}{y+1}\Big]_0^1 dy = \frac{1}{3}\int_0^1 \frac{1}{y+1} \, dy = \Big[\frac{1}{3}\ln(y+1)\Big]_0^1 = \frac{1}{3}(\ln 2 - \ln 1) = \boxed{\frac{1}{3}\ln 2}$

   (c) $\int\int\int_E e^{\frac{z}{y}} \, dV$, where $E = \{(x, y, z) : 0 \le y \le 1, y \le x \le 1, 0 \le z \le xy\}$.

   $\int\int\int_E e^{\frac{z}{y}} \, dV = \int_0^1 \int_y^1 \int_0^{xy} e^{\frac{z}{y}} \, dz \, dx \, dy = \int_0^1 \int_y^1 \Big[ye^{\frac{z}{y}}\Big]_0^{xy} dx \, dy$

   $= \int_0^1 \int_y^1 (ye^x - y) \, dx \, dy = \int_0^1 \Big[ye^x - xy\Big]_y^1 dy = \int_0^1 (ey - y - ye^y + y^2) \, dy$

   $= \Big[\frac{1}{2}ey^2 - \frac{1}{2}y^2 - (y-1)e^y + \frac{1}{3}y^3\Big]_0^1$ [$\int ye^y \, dy$ found by integrating by parts]

   $= \frac{1}{2}e - \frac{1}{2} + \frac{1}{3} - 1 = \boxed{\frac{1}{2}e - \frac{7}{6}}$

(d) $\iiint_E \sin y \, dV$, where E lies below the plane $z = x$ and above the triangular region with vertices $(0,0,0)$, $(\pi,0,0)$, and $(0,\pi,0)$.

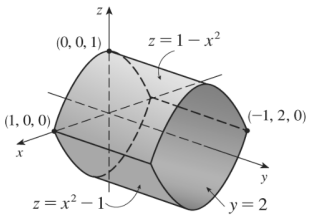
$$\iiint_E \sin y \, dV = \int_0^\pi \int_0^{\pi-x} \int_0^x \sin y \, dz \, dy \, dx = \int_0^\pi \int_0^{\pi-x} \Big[z \sin y \Big]_0^x dy \, dx$$

$$= \int_0^\pi \int_0^{\pi-x} x \sin y \, dy \, dx = \int_0^\pi \Big[-x \cos y \Big]_0^{\pi-x} dx = \int_0^\pi [-x\cos(\pi - x) + x] \, dx$$

$$= \Big[x \sin(\pi - x) - \cos(\pi - x) + \tfrac{1}{2}x^2 \Big]_0^\pi \quad \text{[integration by parts]}$$

$$= 0 - 1 + \tfrac{1}{2}\pi^2 - 0 - 1 - 0 = \boxed{\frac{1}{2}\pi^2 - 2}$$

(e) $\iiint_E (x - y) \, dV$, where E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$.



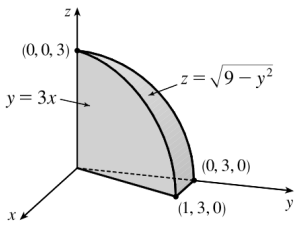
$$\iiint_E (x - y) \, dV = \int_{-1}^1 \int_0^2 \int_{x^2-1}^{1-x^2} (x - y) \, dz \, dy \, dx = \int_{-1}^1 \int_0^2 (x - y) \cdot z \, dz \, dy \, dx$$

$$= \int_{-1}^1 \int_0^2 (x - y)(1 - x^2 - (x^2 - 1)) \, dy \, dx = \int_{-1}^1 \int_0^2 (2x - 2x^3 - 2y + 2x^2 y) \, dy \, dx$$

$$= \int_{-1}^1 \Big[2xy - 2x^3 y - y^2 + x^2 y^2 \Big]_0^2 dx = \int_{-1}^1 (4x - 4x^3 - 4 + 4x^2) \, dx$$

$$= \Big[2x^2 - x^4 - 4x + \tfrac{4}{3}x^3 \Big]_{-1}^1 = -\tfrac{5}{3} - \tfrac{11}{3} = \boxed{-\frac{16}{3}}$$

(f) $\int\int\int_E z\,dV$, where E is bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $y=3x$, and $z=0$ in the first octant.



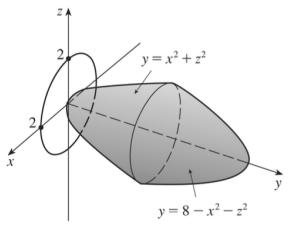
$$\int\int\int_E z\,dV=\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx=\int_0^1\int_{3x}^3 \tfrac{1}{2}z^2\,dy\,dx=\int_0^1\int_{3x}^3 \tfrac{1}{2}(9-y^2)\,dy\,dx$$

$$=\int_0^1\left[\tfrac{9}{2}y-\tfrac{1}{6}y^3\right]_{3x}^3 dx=\int_0^1\left[9-\tfrac{27}{2}x+\tfrac{9}{2}x^3\right]dx=\left[9x-\tfrac{27}{4}x^2+\tfrac{9}{8}x^4\right]_0^1$$

$$=9-\tfrac{27}{4}+\tfrac{9}{8}=\boxed{\frac{27}{8}}$$

2. Use a triple integral to find the volume of the given solid.

(a) The solid enclosed by the paraboloids $y=x^2+z^2$ and $y=8-x^2-z^2$.



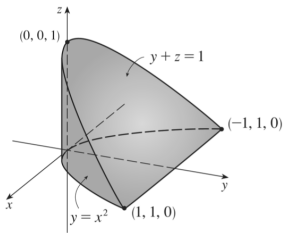
The paraboloids intersect when $x^2+z^2=8-x^2-z^2$ implies $x^2+z^2=4$, so the intersection is the circle $x^2+z^2=4$, $y=4$. The projection of e onto the xz-plane is the disk $x^2+y^2\le 4$, so $E=\{(x,y,z): x^2+z^2\le y\le 8-x^2-z^2, x^2+z^2\le 4\}$. Let $D=\{(x,z): x^2+z^2\le 4\}$. Then using polar coordinates $x=r\cos\theta$ and $z=r\sin\theta$, we have

$$V=\int\int\int_E dV=\int\int_D\left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right)dA=\int\int_D(8-2x^2-2z^2)\,dA$$

$= \int_0^{2\pi} \int_0^2 (8 - r^2) r \, dr \, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 (8r - 2r^3) \, dr = \Big[\theta\Big]_0^{2\pi} \cdot \Big[4r^2 - \frac{1}{2}r^4\Big]_0^2$

$= 2\pi(16 - 8) = \boxed{16\pi}$

(b) The solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.



The plane $y + z = 1$ intersects the xy-plane in the line $y = 1$, so $E = \{(x, y, z) : -1 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1 - y\}$ and

$V = \int\int\int_E dV = \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz \, dy \, dx = \int_{-1}^1 \int_{x^2}^1 \Big[z\Big]_0^{1-y} dy \, dx$

$= \int_{-1}^1 \int_{x^2}^1 (1 - y) \, dy \, dx = \int_{-1}^1 \Big[y - \frac{y^2}{2}\Big]_{x^2}^1 dx \int_{-1}^1 \Big(\frac{1}{2} - x^2 + \frac{1}{2}x^4\Big) dx$

$= \Big[\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5\Big]_{-1}^1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \boxed{\frac{8}{15}}$

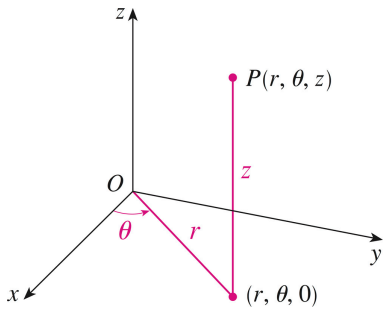
D.5 Triple Integrals in Cylindrical Coordinates

In this section we consider another way to define a point in three-dimensional space, analog to polar coordinates in two dimensions.

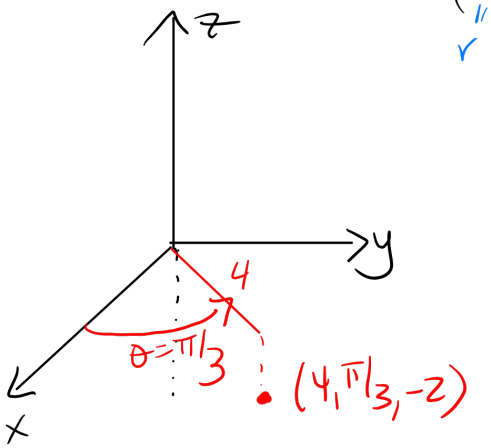
Definition. In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

Note. To convert a point (x, y, z) to cylindrical coordinates: $\boxed{x = r\cos\theta, y = r\sin\theta, z = z}$

To convert a point (r, θ, z) to rectangular coordinates: $\boxed{r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z}$



Example. Plot the point $\left(4, \frac{\pi}{3}, -2\right)$. Then, convert it to rectangular coordinates.



$r = 4$, $\theta = \frac{\pi}{3}$, $z = -2$

$z = -2$

$x = r\cos\theta = 4\cos\frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2$

$y = r\sin\theta = 4\sin\frac{\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$

In rectangular coordinates the point is

$(2, 2\sqrt{3}, -2)$.

Example. Convert the point $(-1, 1, 1)$ to cylindrical coordinates

Given $x = -1, y = 1, z = 1$. $r^2 = x^2 + y^2 = 1 + 1 = 2 \Rightarrow r = \sqrt{2}$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = -1 \Rightarrow \theta = \frac{3\pi}{4} + 2n\pi$ ($(-1,1)$ lies in the 2nd quadrant)

One set of cylindrical coordinates is $(\sqrt{2}, 3\pi/4, 1)$.

Example. Write the equation $x^2 - x + y^2 + z^2 = 1$ in cylindrical coordinates.

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$r^2\cos^2\theta - r\cos\theta + r^2\sin^2\theta + z^2 = 1$

$r^2 - r\cos\theta + z^2 = 1$

$$z^2 = 1 - r^2 + r\cos\theta$$

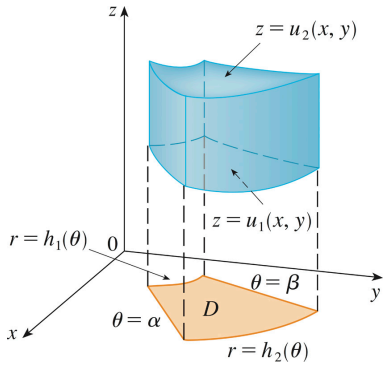
Evaluating Triple Integrals with Cylindrical Coordinates

If E is a **Type I** region whose projection D onto the xy-plane is described in polar coordinates,

$$E = \{(x, y, z) : (x, y) \text{ in } D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$



then we can integrate as follows:

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Note. It is convenient to use this formula when E is a solid region easily described in cylindrical coordinates, especially if $f(x, y, z)$ involves the expression $x^2 + y^2$.

Example. Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

$r^2=16$

$E=\{(r,\theta,z) \mid 0\le r\le 4,\ 0\le\theta\le 2\pi,\ -5\le z\le 4\}$

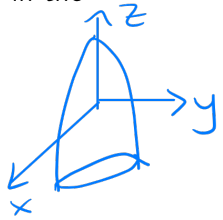
$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz = \theta\Big]_0^{2\pi}\cdot\frac{r^3}{3}\Big]_0^4\cdot z\Big]_{-5}^4 = 2\pi\cdot\frac{64}{3}\cdot(9)$$

$$= \boxed{384\pi}$$

Example. Use cylindrical coordinates to evaluate $\iiint_E (x+y+z)\,dV$ where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.

$z=4-r^2$



The paraboloid intersects the xy-plane in the circle $x^2+y^2=4 \Rightarrow r^2=4 \Rightarrow r=2$

$E=\{(r,\theta,z) \mid 0\le r\le 2,\ 0\le\theta\le\pi/2,\ 0\le z\le 4-r^2\}$

$$\iiint_E (x+y+z)\,dV = \int_0^{\pi/2}\int_0^2\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\cdot z + r^2\sin\theta\cdot z + r\frac{z^2}{2}\Big]_0^{4-r^2} dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 r^2(\cos\theta+\sin\theta)\cdot(4-r^2) + \frac{r(4-r^2)^2}{2}\,dr\,d\theta$$

$u=4-r^2$, $du=-2r\,dr$

$$= \int_0^{\pi/2} (4r^2-r^4)(\cos\theta+\sin\theta) + \frac{r(4-r^2)^2}{2}\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\left(\frac{4r^3}{3}-\frac{r^5}{5}\right)(\cos\theta+\sin\theta) - \frac{1}{4}\frac{(4-r^2)^3}{3}\right]_0^2 d\theta$$

Example. Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

$$= \int_0^{\pi/2} \left[\left(\frac{32}{3} - \frac{32}{5}\right)(\cos\theta + \sin\theta) - 0 - \left(0 - \frac{1}{4}\cdot\frac{4^3}{3}\right) \right] d\theta$$

$$= \int_0^{\pi/2} \frac{64}{15}(\cos\theta + \sin\theta) + \frac{16}{3}\, d\theta$$

$$= \frac{64}{15}\left(\sin\theta - \cos\theta\right) + \frac{16}{3}\theta \Big]_0^{\pi/2}$$

Example. Use cylindrical coordinates to evaluate $\iiint_E (x+y+z)\,dV$ where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.

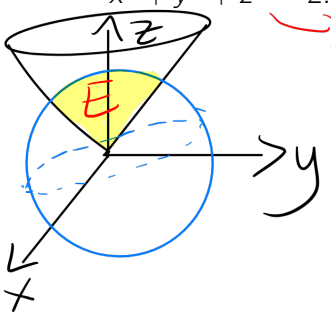
$$= \frac{64}{15}\left(\sin\pi/2 - \cos\pi/2\right) + \frac{16\cdot\pi}{3\cdot 2} - \frac{64}{15}\left(\sin 0 - \cos 0\right) - 0$$

$$= \frac{64}{15} + \frac{8\pi}{3} + \frac{64}{15} = \boxed{\frac{128}{15} + \frac{8\pi}{3}}$$

$z^2 = 2 - r^2$

$z = \sqrt{2-r^2}$

Example. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.



$\rightarrow r^2 + z^2 = 2$

$z = r$

In cylindrical coordinates, E is bounded below by the cone $z = r$, and above by the sphere $r^2 + z^2 = 2$. $\Rightarrow r^2 + r^2 = 2 \Rightarrow 2r^2 = 2 \Rightarrow r = 1$

$$E = \{(r,\theta,z) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi, r \le z \le \sqrt{2-r^2}\}$$

$$V = \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 rz \Big]_r^{\sqrt{2-r^2}} dr \, d\theta = \int_0^{2\pi} \int_0^1 r\sqrt{2-r^2} - r^2 \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\sqrt{2-r^2} \, dr \, d\theta - \int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta$$

$u = 2 - r^2$

$du = -2r \, dr$

$-\frac{1}{2} du = r \, dr$

$$= \int_0^{2\pi} \int_2^1 -\frac{1}{2} \sqrt{u} \, du \, d\theta - \int_0^{2\pi} d\theta \cdot \frac{r^3}{3} \Big]_0^1$$

$$= \int_0^{2\pi} \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big]_1^2 d\theta - 2\pi \cdot \frac{1}{3}$$

$$= \int_0^{2\pi} \frac{1}{3}(2^{3/2} - 1) \, d\theta - 2\pi \cdot \frac{1}{3} = \boxed{\frac{1}{3}(2^{3/2} - 1) \cdot 2\pi - \frac{2\pi}{3}}$$

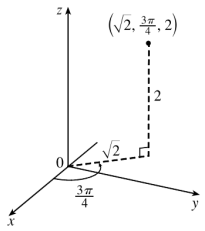
Homework D.5 - Triple Integrals in Cylindrical Coordinates

Prof. Leyda Almodóvar

Name: _____

1. Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

   (a) $\left(\sqrt{2}, \frac{3\pi}{4}, 2\right)$

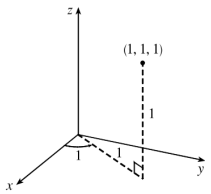


   $x = \sqrt{2}\cos\frac{3\pi}{4} = \sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -1,$

   $y = \sqrt{2}\sin\frac{3\pi}{4} = \sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 1,\ z = 2$

   so the point is $(-1, 1, 2)$ in rectangular coordinates.

   (b) $(1, 1, 1)$



   $x = 1\cos 1 = \cos 1,\ y = 1\sin 1 = \sin 1,\ z = 1,$ so the point is $(\cos 1, \sin 1, 1)$ in rectangular coordinates.

2. Change from rectangular to cylindrical coordinates.

   (a) $(-\sqrt{2}, \sqrt{2}, 1)$

   $r^2 = (-\sqrt{2})^2 + (\sqrt{2})^2 = 4$. so $r = 2$; $\tan\theta = \frac{\sqrt{2}}{-\sqrt{2}} = -1$ and the point $(-\sqrt{2}, \sqrt{2})$ is the second quadrant of the xy-plane, so $\theta = \frac{3\pi}{4} + 2n\pi$; $z = 1$. Hence, one set of cylindrical coordinates is $\left(2, \frac{3\pi}{4}, 1\right)$.

   (b) $(2, 2, 2)$

   $r^2 = 2^2 + 2^2 = 8$, so $r = \sqrt{8} = 2\sqrt{2}$; $\tan\theta = \frac{2}{2} = 1$ and the point $(2, 2)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4} + 2n\pi$; $z = 2$. Hence, one set of cylindrical coordinates is $\left(2\sqrt{2}, \frac{\pi}{4}, 2\right)$.

3. Write the equations in cylindrical coordinates.

   (a) $2x^2 + 2y^2 - z^2 = 4$

   The equation $2x^2 + 2y^2 - z^2 = 4$ can be written as $2(x^2 + y^2) - z^2 = 4$ which becomes $2r^2 - z^2 = 4$ or $z^2 = 2r^2 - 4$ in cylindrical coordinates.

   (b) $2x - y + z = 1$

   Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $2x - y + z = 1$ becomes $2r\cos\theta - r\sin\theta + z = 1$ or $z = 1 + r(\sin\theta - 2\cos\theta)$ in cylindrical coordinates.

4. Evaluate the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$ by changing to cylindrical coordinates.

   The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9-x^2}$ which describes the upper half of a circle of radius 3 in the xy-plane centered at $(0, 0)$. Hence,

   $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,d = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2}\sqrt{r^2}r\,dz\,dr\,d\theta$$

   $$= \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{9-r^2} r^2\,dz\,dr\,d\theta = \int_{0}^{\pi}\int_{0}^{3} r^2(9-r^2)\,dr\,d\theta = \int_{0}^{\pi} d\theta \int_{0}^{3}(9r^2 - r^4)\,dr$$

   $$= \Big[\theta\Big]_0^{\pi}\Big[3r^3 - \tfrac{1}{5}r^5\Big]_0^{3} = \pi\left(81 - \tfrac{243}{5}\right) = \boxed{\frac{162}{5}\pi}$$

5. Use cylindrical coordinates to evaluate $\int\int\int_E z\,dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

The paraboloid $z = x^2 + y^2 = r^2$ intersects the plane $z = 4$ in the circle $x^2 + y^2 = 4$ or $r^2 = 4$, which implies $r = 2$, so in cylindrical coordinates, E is given by $E = \{r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r \le 2, r^2 \le z \le 4\}$. Hence,

$$\int\int\int_E z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 (z) r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\tfrac{1}{2}rz^2\right]_{r^2}^4 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 \left(8r - \tfrac{1}{2}r^5\right) dr = 2\pi\left[4r^2 - \tfrac{1}{12}r^6\right]_0^2$$

$$= 2\pi\left(16 - \tfrac{16}{3}\right) = \boxed{\frac{64}{3}\pi}$$

6. Use cylindrical coordinates to evaluate $\int\int\int_E (x - y)\,dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.

In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\sin\theta + 4$ and the cylinders $r = 1$ and $r = 4$, so E is given by $E = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 1 \le r \le 4, 0 \le z \le r\sin\theta + 4\}$. Hence,

$$\int\int\int_E (x - y)\,dV = \int_0^{2\pi}\int_1^4\int_0^{r\sin\theta+4} (r\cos\theta - r\sin\theta) r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 (r^2\cos\theta - r^2\sin\theta)\left[z\right]_0^{r\sin\theta+4} dr\,d\theta = \int_0^{2\pi}\int_1^4 (r^2\cos\theta - r^2\sin\theta)(r\sin\theta + 4)\,dr\,d\theta$$

$$\int_0^{2\pi}\int_1^4 [(r^3(\sin\theta\cos\theta - \sin^2\theta) + 4r^2(\cos\theta - \sin\theta)]\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\tfrac{1}{4}r^4(\sin\theta\cos\theta - \sin^2\theta) + \tfrac{4}{3}r^3(\cos\theta - \sin\theta)\right]_1^4 d\theta$$

$$= \int_0^{2\pi}\left[\left(64 - \tfrac{1}{4}\right)(\sin\theta\cos\theta - \sin^2\theta) + \left(\tfrac{256}{3} - \tfrac{4}{3}\right)(\cos\theta - \sin\theta)\right] d\theta$$

$$= \int_0^{2\pi}\left[\left(\tfrac{255}{4}\right)(\sin\theta\cos\theta - \sin^2\theta) + 84(\cos\theta - \sin\theta)\right] d\theta$$

$$= \left[\tfrac{255}{4}\left(\tfrac{1}{2}\sin^2\theta - \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right) + 84(\sin\theta + \cos\theta)\right]_0^{2\pi}$$

$$= \tfrac{255}{4}(-\pi) + 84(1) - 0 - 84(1) = \boxed{-\frac{255}{4}}$$

7. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$, and the sphere $x^2 + y^2 + z^2 = 4$.

In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so
$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}\}$. Hence, the volume is

$$\iiint_E \mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \int_0^1 \Big[rz\Big]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \mathrm{d}r \,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^1 2r\sqrt{4 - r^2} \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 2r\sqrt{4 - r^2} \,\mathrm{d}r = 2\pi \Big[-\tfrac{2}{3}(4-r^2)^{\frac{3}{2}}\Big]_0^1 = \boxed{\frac{4}{3}\pi(8 - 3^{\frac{3}{2}})}$$

8. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

In cylindrical coordinates E is bounded below by the paraboloid $z = r^2$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2 - r^2}$. The paraboloid and the sphere intersect when $r^2 + r^4 = 2$, which implies $(r^2 + 2)(r^2 - 1) = 0$. Hence, $r = 1$, so
$E = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r \le 1, r^2 \le z \le \sqrt{2 - r^2}\}$ and the volume is

$$\iiint_E \mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \int_0^1 \Big[rz\Big]_{r^2}^{\sqrt{2-r^2}} \mathrm{d}r \,\mathrm{d}\theta$$

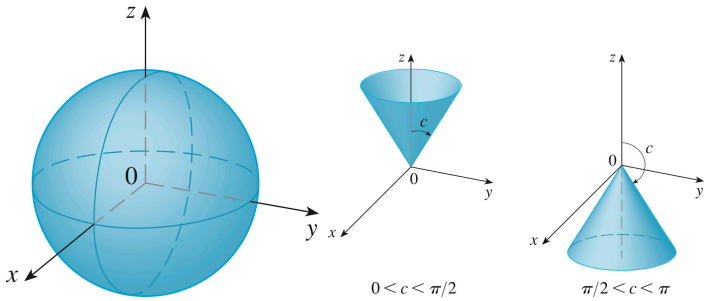
$$= \int_0^{2\pi} \int_0^1 (r\sqrt{2 - r^2} - r^3) \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 (r\sqrt{2 - r^2} - r^3) \,\mathrm{d}r$$

$$= 2\pi \Big[-\tfrac{1}{3}(2 - r^2)^{\frac{3}{2}} - \tfrac{1}{4}r^4 \Big]_0^1 = 2\pi \Big(-\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{3} \cdot 2^{\frac{3}{2}} - 0 \Big)$$

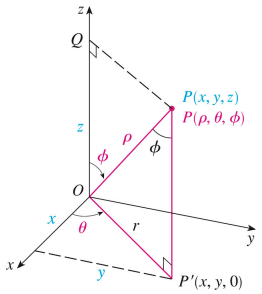
$$= 2\pi \Big(-\tfrac{7}{12} + \tfrac{2}{3}\sqrt{2} \Big) = \boxed{\Big(-\frac{7}{6} + \frac{4}{3}\sqrt{2} \Big)\pi}$$

D.6 Triple Integrals in Spherical Coordinates

In this section we introduce a coordinate system that simplifies the evaluation of triple integrals over regions bounded by spheres or cones due to the symmetry of these surfaces.



Definition. In the **spherical coordinate system**, a point P in three-dimensional space is represented by (ρ, θ, ϕ), where $\rho = |OP|$ is the distance from the origin O to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment.



Note.

- $\rho \geq 0$, $0 \leq \phi \leq \pi$
- To convert to spherical coordinates: $\boxed{x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi}$
- To convert to rectangular coordinates: $\boxed{\rho^2 = x^2 + y^2 + z^2}$
- $\rho = R$ defines a sphere of radius R
- If the solid is a ball, $\boxed{B = \{(\rho, \theta, \phi) | 0 \leq \rho \leq R, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}}$
- If $\phi = 0$, points are on the positive z-axis
- If $\phi = \pi$, points are on the negative z-axis
- If $\phi = \dfrac{\pi}{2}$, points are on the xy-plane
- For $0 < c < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < c < \pi$, the equation $\phi = c$ defines a nappe of a cone

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Evaluating Triple Integrals with Spherical Coordinates

Let E be a spherical wedge given by $E = \{(, \theta, \phi) : a \le \rho \le b, \alpha \le \beta, c \le \phi \le d\}$. Then,

$$\int\int\int_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Example. Evaluate $\int\int\int_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV$, where $B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$.

$B = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

$= \int_0^\pi \sin\phi\,d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 e^{\rho^3}\rho^2\,d\rho$ $\quad u = \rho^3$, $du = 3\rho^2 d\rho$, $\frac{1}{3}du = \rho^2 d\rho$

$= [-\cos\phi]_0^\pi \cdot [\theta]_0^{2\pi} \cdot \left[\frac{1}{3}e^{\rho^3}\right]_0^1$

$= (1+1)(2\pi-0)\cdot\left(\frac{1}{3}e - \frac{1}{3}\right) = \boxed{\frac{4\pi}{3}(e-1)}$

Example. Evaluate $\int\int\int_B (x^2+y^2+z^2)^2\,dV$, where B is the ball with center the origin and radius 5.

$B = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 5,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$

$\iiint_B (x^2+y^2+z^2)^2\,dV = \int_0^\pi \int_0^{2\pi} \int_0^5 (\rho^2)^2 \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

$= \int_0^\pi \sin\phi\,d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^5 \rho^6\,d\rho$

$= [-\cos\phi]_0^\pi \cdot [\theta]_0^{2\pi} \cdot \left[\frac{\rho^7}{7}\right]_0^5 = 2 \cdot 2\pi \cdot \frac{5^7}{7} = \boxed{\frac{312,500\pi}{7}}$

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Example. Evaluate $\iiint_E (x^2+y^2)\,dV$, where E lies between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$$E=\{(\rho,\theta,\phi)\mid 2\le\rho\le 3,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi\}$$

$$\iiint_E (x^2+y^2)\,dV$$

$$=\int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi}\int_0^{2\pi}\int_2^3 (\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi}\sin^3\phi\,d\phi\cdot\int_0^{2\pi}d\theta\cdot\int_2^3\rho^4\,d\rho$$

$$=\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi\cdot[\theta]_0^{2\pi}\cdot\left[\frac{\rho^5}{5}\right]_2^3$$

$$=-\int_1^{-1}(1-u^2)\,du\cdot[2\pi]\cdot\left[\frac{3^5}{5}-\frac{2^5}{5}\right]$$

$u=\cos\phi$
$du=-\sin\phi\,d\phi$

$$=\left[u-\frac{u^3}{3}\right]_{-1}^{1}\cdot(2\pi)\cdot\left(\frac{243}{5}-\frac{32}{5}\right)$$

$$=\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]\cdot(2\pi)\left(\frac{211}{5}\right)$$

$$=\boxed{\frac{1688\pi}{15}}$$

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Homework D.6 - Triple Integrals in Spherical Coordinates

Prof. Leyda Almodóvar

Name: _____

1. Use spherical coordinates to evaluate the integral $\int\int\int_E y^2z^2\,\mathrm{d}V$, where E lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 1$.

   In spherical coordinates, E is represented by

   $$E = \left\{(\rho,\theta,\phi) : 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{3}\right\}$$

   Hence,

   $$\int\int\int_E y^2z^2\,\mathrm{d}V = \int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^1 (\rho\sin\phi\sin\theta)^2(\rho\cos\phi)^2\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$$

   $$= \int_0^{\frac{\pi}{3}} \sin^3\phi\cos^2\phi\,\mathrm{d}\phi \int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta \int_0^1 \rho^6\,\mathrm{d}\rho$$

   $$= \int_0^{\frac{\pi}{3}} (1-\cos^2\phi)\cos^2\phi\sin\phi\,\mathrm{d}\phi \int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\,\mathrm{d}\theta \int_0^1 \rho^6\,\mathrm{d}\rho$$

   $$= \left[\tfrac{1}{5}\cos^5\phi - \tfrac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{3}}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{7}\rho^7\right]_0^1$$

   $$= \left[\tfrac{1}{5}\left(\tfrac{1}{2}\right)^5 - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - \tfrac{1}{5} + \tfrac{1}{3}\right](\pi - 0)\left(\tfrac{1}{7} - 0\right) = \tfrac{47}{480}\cdot\pi\cdot\tfrac{1}{7} = \boxed{\frac{47}{3360}\pi}$$

2. Use spherical coordinates to evaluate the integral $\int\int\int_E y^2\,\mathrm{d}V$, where E is the solid hemisphere $x^2 + y^2 + z^2 \le 9$, $y \ge 0$.

   In spherical coordinates, E is represented by

   $$E = \left\{(\rho,\theta,\phi) : 0 \le \rho \le 3, 0 \le \theta \le \pi, 0 \le \phi \le \pi\right\}$$

   Hence,

   $$\int\int\int_E y^2\,\mathrm{d}V = \int_0^{\pi}\int_0^{\pi}\int_0^3 (\rho\sin\phi\sin\theta)^2\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{\pi}\sin^3\phi\,\mathrm{d}\phi\int_0^{\pi}\sin^2\theta\int_0^3\rho^4\,\mathrm{d}\rho$$

$= \int_0^{\pi} (1 - \cos^2\phi) \sin\phi \, \mathrm{d}\phi \int_0^{\pi} \frac{1}{2}(1 - \cos 2\theta) \, \mathrm{d}\theta \int_0^3 \rho^4 \, \mathrm{d}\rho$

$= \Big[-\cos\phi + \frac{1}{3}\cos^3\phi \Big]_0^{\pi} \Big[\frac{1}{2}\Big(\theta - \frac{1}{2}\sin 2\theta\Big) \Big]_0^{\pi} \Big[\frac{1}{5}\rho^5 \Big]_0^3$

$= \Big(\frac{2}{3} + \frac{2}{3}\Big)\Big(\frac{1}{2}\pi\Big)\Big(\frac{1}{2}(243)\Big) = \boxed{\frac{162\pi}{5}}$

3. Use spherical coordinates to evaluate the integral $\int\int\int_E \sqrt{x^2 + y^2 + z^2} \, \mathrm{d}V$, where E lies above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

The cone $z = \sqrt{x^2 + y^2}$ is equivalent to $\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta}$, which implies $\rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$. Hence, $\cos\phi = \sin\phi$, which implies $\phi = \frac{\pi}{4}$ and E is represented by

$$E = \Big\{ (\rho, \theta, \phi) : 1 \le \rho \le 2, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4} \Big\}$$

Also, $\sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2} = \rho$, so

$\int\int\int_E \sqrt{x^2 + y^2 + z^2} \, \mathrm{d}V = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_1^2 \rho \cdot \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi = \int_0^{\frac{\pi}{4}} \sin\phi \, \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta \int_1^2 \rho^3 \, \mathrm{d}\rho$

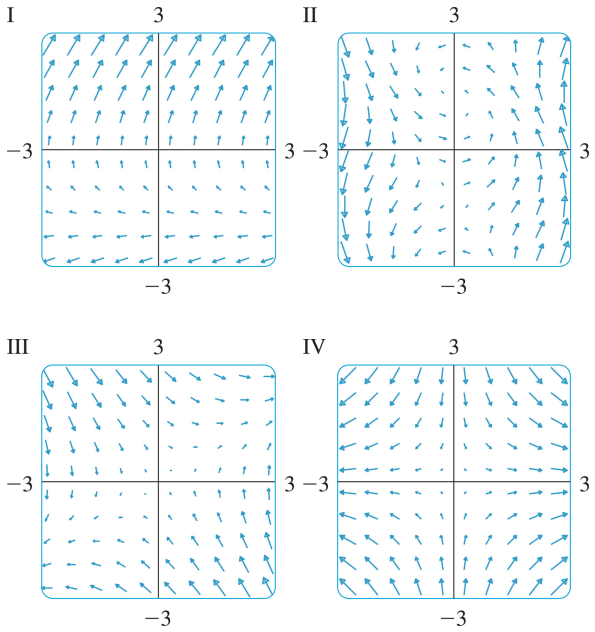
$= \Big[-\cos\phi \Big]_0^{\frac{\pi}{4}} \Big[\theta\Big]_0^{2\pi} \Big[\frac{1}{4}\rho^4\Big]_1^2 = \Big(-\frac{\sqrt{2}}{2} + 1 \Big)(2\pi) \cdot \frac{1}{4}(16 - 1) = \boxed{\frac{15}{2}\pi\Big(1 - \frac{\sqrt{2}}{2}\Big)}$

Homework E.1 - Vector Fields

Prof. Leyda Almodóvar

Name: _____

1. Match the vector fields F with the plots labeled I-IV. Give reasons for your choices.



(a) $\vec{F}(x, y) = \langle x, -y \rangle$

$\vec{F}(x, y) = \langle x, -y \rangle$ corresponds to graph IV. In the first quadrant all the vector have positive x-components and negative y-components, in the second quadrant all vectors have negative x- and y-components, in the third quadrant all vectors have negative x-components and positive y-components, and in the fourth quadrant all vectors have positive x- and y- components. In addition, the vectors get shorter as we approach the origin.

(b) $\vec{F}(x,y) = \langle y, x-y \rangle$

$\vec{F}(x,y) = \langle y, x-y \rangle$ corresponds to graph III. All vectors in quadrants I and II have positive x-components while all vectors in quadrants III and IV have negative x-components. In addition, vectors along the line $y = x$ are horizontal, and vectors get shorter as we approach the origin.

(c) $\vec{F}(x,y) = \langle y, y+2 \rangle$

$\vec{F}(x,y) = \langle y, y+2 \rangle$ corresponds to graph I. All vectors in quadrants I and II have positive x-components while all vectors in quadrants III and IV have negative x-components. Vectors along the line $y = -2$ are horizontal, and the vectors are independent of x (vectors along horizontal lines are identical).

(d) $\vec{F}(x,y) = \langle \cos(x+y), x \rangle$

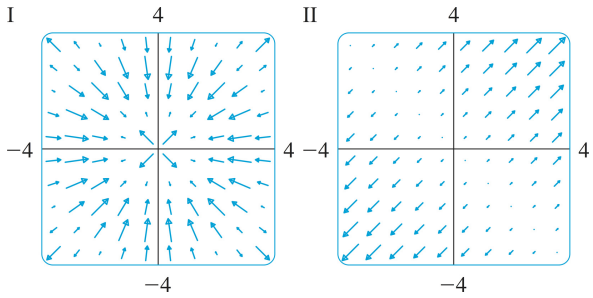
$\vec{F}(x,y) = \langle \cos(x+y), x \rangle$ corresponds to graph II. All vectors in quadrants I and IV have positive y-components while all vectors in quadrants II and III have negative y-components. Also, the y-components of vectors along any vertical line remain constant while the x-component oscillates.

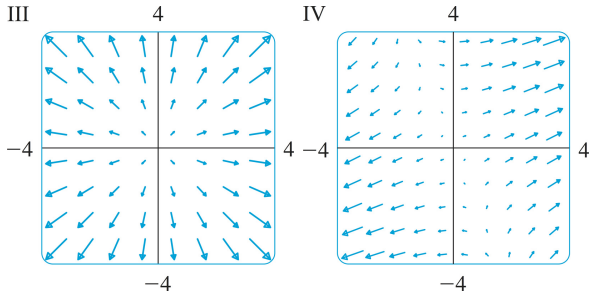
2. Find the gradient vector field of $f(x,y) = y\sin(xy)$.

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle y\cos(xy)\cdot y, [y\cdot x\cos(xy) + \sin(xy)\cdot 1] \rangle$

$= \boxed{\langle y^2\cos(xy), [xy\cos(xy) + \sin(xy)] \rangle}$

3. Match the functions f with the plots of their gradient vector fields labeled I-IV. Give reasons for your choices.





(a) $f(x,y) = x^2 + y^2$

$\nabla f(x,y) = \langle 2x, 2y \rangle$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph III.

(b) $f(x,y) = x(x+y)$

$f(x,y) = x(x+y) = x^2 + xy$ implies $\nabla f(x,y) = \langle 2x + y, x \rangle$. The y-component of each vector is x, so the vectors point upward in quadrants I and IV and downward in quadrants II and III. Also, the x-component of each vector is 0 along the line $y = -2x$ so the vectors are vertical there. Thus, ∇f is graph IV.

(c) $f(x,y) = (x+y)^2$

$\nabla f(x,y) = \langle 2(x+y), 2(x+y) \rangle$. The x- and y- components of each vector are equal, so all vectors are parallel to the line $y = x$. The vectors are $\vec{0}$ along the line $y = -x$ and their length increases as the distance from this line increases. Thus, ∇f is graph II.

(d) $f(x,y) = \sin\sqrt{x^2+y^2}$

$$\nabla f(x,y) = \left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x)\right]\vec{i} + \left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x)\right]\vec{j}$$

$$= \left\langle \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}x, \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}y \right\rangle$$

Thus each vector is a scalar multiple of its position vector, so the vectors point toward or away from the origin with length that changes in a periodic fashion as we move away from the origin. Thus, ∇f is graph I.

E.2 Line Integrals

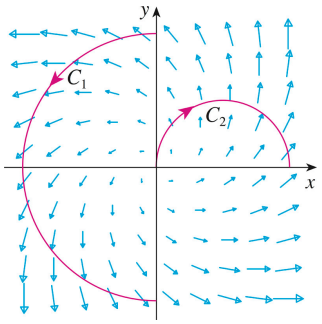
Line Integrals in Space

Definition. A curve is said to be **oriented** if we have chosen a direction to travel along it.

We desire to study flows of vector fields in relation to a given oriented curve. We can consider:

- **Flow along** a curve (how much the vector field flows with or against the curve).
- **Flow across** a curve (how much the vector field flows into or out of the curve).

Vectors starting at C_1 roughly move in the direction of the curve
$\Rightarrow$ vector field flows with C_1



None of the vectors starting at C_2 point in the same direction as the curve
$\Rightarrow$ vector field flows against C_2

To calculate flow along, we can use **line integrals**.

Definition. A **line integral** is an integral of a function calculated along a given curve.

$\rightarrow$ Integrating over a curve instead of integrating over $[a,b]$

Note.

- Simple integrals like $\int_a^b f(x)\,dx$ are also line integrals, they are just a specific type of line integral in which the line is a segment of the x axis, $[a,b]$. You have *always* calculated line integrals in one variable.
- With line integrals we will start with integrating the function $f(x,y)$, a function of two variables, and the values of x and y that we're going to use will be the points (x,y) that lie on a curve C.
- We need to parameterize the curve C: $x = g(t)$ and $y = h(t)$ for $a \le t \le b$. At times we will work with the parameterization of the curve as a vector function. In that case the curve is given by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j}$, $a \le t \le b$.
- A curve is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$ for all t.

Parametric Equations

Definition. The **standard set of parametric equations for a line in 2-space** is given by:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}$$

$m = \frac{b}{a}$ is the slope of the line

Where the line is parallel to $a\vec{i} + b\vec{j}$, the "direction vector," and $t = 0$ gives a point that the line passes through (x_0, y_0).

Definition. To parameterize the line segment from (x_0, y_0) to (x_1, y_1) with $0 \leq t \leq 1$:

$$\begin{aligned} x &= (1-t)x_0 + tx_1 \\ y &= (1-t)y_0 + ty_1 \end{aligned}$$

*Need two points on the line.

Definition. The **parametric equations for a circle in 2-space** of radius r centered at (x_0, y_0) in the xy-plane, with counterclockwise direction, is given by:

$$\begin{aligned} x &= r\cos t + x_0 \\ y &= r\sin t + y_0 \end{aligned}$$

$0 \leq t \leq 2\pi$ (for complete circle)

Note. To make circles clockwise instead of counterclockwise, plug in $-t$, obtaining:

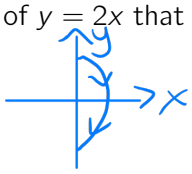
$$\begin{aligned} x &= r\cos(-t) + x_0 = r\cos t + x_0 \\ y &= r\sin(-t) + y_0 = -r\sin t + y_0 \end{aligned}$$

$\cos(-t) = \cos t$

$\sin(-t) = -\sin t$

Note. To obtain only a portion of a parameterized curve, such as a semi or quarter circle, specify the range for t that traces out this part of the curve. For example, for an upper semi-circle, you would specify $0 \leq t \leq \pi$ (think Unit Circle angles). For the line segment of $y = 2x$ that goes from $(0, 0)$ to $(3, 6)$, specify $0 \leq t \leq 3$.

$-\pi/2 \leq t \leq \pi/2$



$\pi/2 \leq t \leq 3\pi/2$

Definition. If f is defined on a smooth curve C, then the line integral of f along C is

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

arc length

$a \leq t \leq b$

$(x-h)^2+(y-k)^2=r^2$ (h,k) is the center → center is $(0,0)=(x_0,y_0)$

Example. Evaluate $\int_C xy^2\,ds$ where C is the right half of the circle $x^2+y^2=16$ traced out in a counterclockwise direction.

First we need to parameterize C: $x=r\cos t+x_0=4\cos t$, $y=r\sin t+x_0=4\sin t$, $-\pi/2\le t\le\pi/2$

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{(-4\sin t)^2+(4\cos t)^2}=\sqrt{16\sin^2t+16\cos^2t}=\sqrt{16}=4$

$\int_C xy^2\,ds=\int_{-\pi/2}^{\pi/2}4\cos t\cdot 16\sin^2t\cdot 4\,dt=256\int_{-\pi/2}^{\pi/2}\cos t\cdot\sin^2t\,dt$

$u=\sin t$, $du=\cos t\,dt$

$=256\cdot\int_{-1}^{1}u^2\,du=256\cdot\frac{u^3}{3}\Big]_{-1}^{1}=256\cdot\frac{2}{3}=\boxed{\frac{512}{3}}$

Example. Evaluate $\int_C 3x^2\,ds$ where C is the line segment from $(1,2)$ to $(3,-1)$. ($x_0=1$, $y_0=2$, $x_1=3$, $y_1=-1$)

$x(t)=(1-t)x_0+tx_1=(1-t)\cdot 1+t\cdot 3=2t+1$, $0\le t\le 1$

$y(t)=(1-t)y_0+ty_1=(1-t)\cdot 2+t\cdot(-1)=-3t+2$

$x'(t)=2$, $y'(t)=-3$ $\Rightarrow$ $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{2^2+(-3)^2}\,dt=\sqrt{13}\,dt$

$\int_C 3x^2\,ds=\int_0^1 3(2t+1)^2\cdot\sqrt{13}\,dt=3\sqrt{13}\int_0^1 4t^2+4t+1\,dt$

$=3\sqrt{13}\cdot\left[\frac{4t^3}{3}+2t^2+t\right]_0^1=3\sqrt{13}\left[\frac{4}{3}+2+1\right]=3\sqrt{13}\left[\frac{4+9}{3}\right]$

$=\boxed{13\sqrt{13}}$

Example. Evaluate $\int_C x\,ds$ for $C: y=x^2, -1\le x\le 1$.

$x=t\Rightarrow y=t^2$, $-1\le t\le 1$

$\frac{dx}{dt}=1$, $\frac{dy}{dt}=2t$ $\Rightarrow$ $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{1+4t^2}\,dt$

$\int_C x\,ds=\int_{-1}^{1}t\sqrt{1+4t^2}\,dt=\frac{1}{8}\int_5\sqrt{u}\,du=\boxed{0}$

$u=1+4t^2$

$du=8t\,dt$

$\frac{1}{8}du=t\,dt$

Note: $\int_{-1}^{1}x\,dx=0$

Line Integrals of Vector Fields

Definition. The **flow of a vector field** $\vec{F}(\vec{r})$ along a given oriented curve C is given by the line integral:

$$\int_C \vec{F}\, d\vec{r}$$

- Flow along also represents work done by the force $\vec{F}(\vec{r})$ along the curve C. Example: Work done by a gravity vector field on a particle as it moves along a path.
- Flow along is also called the **circulation** of $\vec{F}$ around C if C is a closed curve.

initial point = terminal point

Computing Line Integrals

Formula and algorithm to compute the flow of vector field along a curve:

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \le t \le b$, is a smooth parameterization of an oriented curve C and $\vec{F}$ is a vector field which is continuous on C, then,

$$\int_C \vec{F}\, d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

dot product

To compute the flow of a vector field along a curve, perform the following steps:

1. Parameterize the curve C to get $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \le t \le b$.
2. In the formula for $\vec{F}$, substitute $x = x(t)$ and $y = y(t)$ from the parameterization of C done above.
3. Compute the component-wise derivative of $\vec{r}(t)$ to get $\vec{r}\,'(t)$, which will just be $x'(t)\vec{i} + y'(t)\vec{j}$.
4. Compute the dot product of the **vectors** $\vec{F}(\vec{r}(t))$ (found in step 2) and $\vec{r}\,'(t)$ (found in step 3). You should get an expression with only t's in it, no $\vec{i}$'s, $\vec{j}$'s, x's, or y's.
5. Plug your result from step 4 into an integral with a and b ($a \le t \le b$ from step 1) as the bounds, dt on the far right.
6. Integrate with respect to t.

Example. Find the flow along the left half of a circle of radius 2, centered at $(0, 0)$, oriented counterclockwise, of the vector field $\vec{F} = (x + y)\vec{i} + y\vec{j}$.

$x(t) = r\cos t + x_0 = 2\cos t$
$y(t) = r\sin t + y_0 = 2\sin t$
$\pi/2 \le t \le 3\pi/2$
$\Rightarrow \vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j}$
$\vec{r}\,'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$

$\vec{F}(\vec{r}(t)) = (2\cos t + 2\sin t)\vec{i} + 2\sin t\,\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -4\cos t\sin t - 4\sin^2 t + 4\sin t\cos t = -4\sin^2 t$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\pi/2}^{3\pi/2} -4\sin^2 t\, dt = -4\cdot\frac{1}{2}\int_{\pi/2}^{3\pi/2} 1-\cos 2t\, dt = -2\left[t - \frac{\sin(2t)}{2}\right]_{\pi/2}^{3\pi/2}$$

$$= -2\left[3\pi/2 - \frac{\sin 3\pi}{2} - \pi/2 + \frac{\sin\pi}{2}\right] = -2(\pi) = \boxed{-2\pi}$$

vector field flows against the curve

- Clearly, all normal rules of integration apply.
- One additional rule:

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_{C} \vec{F} \cdot d\vec{r}$$

In words, reversing the orientation/direction of the curve C makes the integral negative.

Example. In the previous example, if we want to consider the flow along when the half-circle is oriented **clockwise**, we could just take the negative of the result found in the example.

$\int_{-C} \vec{F}\,d\vec{r} = -(-2\pi) = \boxed{2\pi}$

Example. Compute the flow along the line segment from $(0,0)$ to $(3,3)$ of the gradient vector field for the function $F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

$x(t) = (1-t)x_0 + tx_1 = 3t \quad 0 \le t \le 1$

$y(t) = (1-t)y_0 + ty_1 = 3t$

$\vec{r}(t) = 3t\vec{i} + 3t\vec{j}$

$\vec{r}\,'(t) = 3\vec{i} + 3\vec{j}$

$\nabla F(x,y) = \langle x, y \rangle = x\vec{i} + y\vec{j} = \vec{f}$

$\vec{f}(\vec{r}(t)) = 3t\vec{i} + 3t\vec{j}$

$\vec{f}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 9t + 9t = 18t$

$\int_C \nabla F\,d\vec{r} = \int_0^1 18t\,dt = 9t^2\Big]_0^1 = \boxed{9}$

Example. Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x+y^2)\vec{i} + xz\vec{j} + (y+z)\vec{k}$, $\vec{r}(t) = t^2\vec{i} + t^3\vec{j} - 2t\vec{k}$, $0 \le t \le 2$.

($x(t)\,\vec{i}$, $y(t)\,\vec{j}$, $z(t)\,\vec{k}$)

$\vec{F}(\vec{r}(t)) = (t^2 + t^6)\vec{i} + t^2 \cdot (-2t)\vec{j} + (t^3 - 2t)\vec{k}$

$= (t^2 + t^6)\vec{i} - 2t^3\vec{j} + (t^3 - 2t)\vec{k}$

$\vec{r}\,'(t) = 2t\vec{i} + 3t^2\vec{j} - 2\vec{k}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2t^3 + 2t^7 - 6t^5 - 2t^3 + 4t = 2t^7 - 6t^5 + 4t$

$\int_C \vec{F}\,d\vec{r} = \int_0^2 2t^7 - 6t^5 + 4t\,dt = \frac{2t^8}{8} - t^6 + 2t^2\Big]_0^2$

$= \frac{1}{4} \cdot 2^8 - 2^6 + 2^3 = \boxed{8}$

Homework E.2 - Line Integrals

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the line integral, where C is the given curve.

   (a) $\int_C \frac{x}{y}\,ds$, $C: x = t^3$, $y = t^4$, $1 \le t \le 2$

   $\int_C \frac{x}{y}\,ds = \int_1^2 \frac{t^3}{t^4}\sqrt{(3t^2)^2 + (4t^3)^2}\,dt = \int_1^2 \frac{t^3}{t^4}\sqrt{9t^4 + 16t^6}\,dt = \int_1^2 \frac{1}{t}\cdot t^2\sqrt{9 + 16t^2}\,dt$

   $= \int_1^2 t\sqrt{9 + 16t^2}\,dt = \frac{1}{32}\int_5^{\sqrt{73}} \sqrt{u}\,du = \frac{1}{32}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big]_5^{\sqrt{73}} = \boxed{\frac{1}{48}(73\sqrt{73} - 125)}$

   (b) $\int_C xy^4\,ds$, C is the right half of the circle $x^2 + y^2 = 16$

   Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then,

   $\int_C xy^4\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt$

   $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5\cos t\sin^4 t\sqrt{16(\sin^2 t + \cos^2 t)}\,dt = 4^5\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^4 t\cos t)(4)\,dt$

   $= 4^6\left[\frac{1}{5}\sin^5 t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4^6\cdot\frac{2}{5} = \boxed{\frac{8192}{5}}$

   (c) $\int_C xe^y\,ds$, C is the line segment from $(2, 0)$ to $(5, 4)$

   Parametric equations for C are $x = 2 + 3t$, $y = 4t$, $0 \le t \le 1$. Then,

   $\int_C xe^y\,ds = \int_0^1 (2 + 3t)e^{4t}\sqrt{3^2 + 4^2}\,dt = 5\int_0^1 (2 + 3t)e^{4t}\,dt$

   Integrating by parts with $u = 2 + 3t$ and $dv = e^{4t}\,dt$, $\Rightarrow du = 3\,dt$ and $v = \frac{1}{4}e^{4t}$ gives

   $\int_C xe^y\,ds = 5\left[\frac{1}{4}(2 + 3t)e^{4t} - \frac{3}{16}e^{4t}\right]_0^1 = 5\left[\frac{5}{4}e^4 - \frac{3}{16}e^4 - \frac{1}{2} + \frac{3}{16}\right] = \boxed{\frac{85}{16}e^4 - \frac{25}{16}}$

2. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is given by the vector function $\vec{r}(t)$.

(a) $\vec{F}(x,y) = xy^2\vec{i} - x^2\vec{j}$, $\vec{r}(t) = t^3\vec{i} + t^2\vec{j}$, $0 \le t \le 1$

$\vec{r}(t) = t^3\vec{i} + t^2\vec{j}$, so $\vec{F}(\vec{r}(t)) = (t^3)(t^2)^2\vec{i} - (t^3)^2\vec{j} = t^7\vec{i} - t^6\vec{J}$ and $\vec{r}'(t) = 3t^2\vec{i} + 2t\vec{j}$.

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_0^1 (t^7 \cdot 3t^2 - t^6 \cdot 2t)\, dt = \int_0^1 (3t^9 - 2t^7\, dt)$

$= \left[\frac{3}{10}t^{10} - \frac{1}{4}t^8\right]_0^1 = \frac{3}{10} - \frac{1}{4} = \boxed{\frac{1}{20}}$

(b) $\vec{F}(x,y,z) = \sin x\vec{i} + \cos y\vec{j} + xz\vec{k}$, $\vec{r}(t) = t^3\vec{i} - t^2\vec{j} + t\vec{k}$, $0 \le t \le 1$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle \sin t^3, \cos(-t^2), t^4\rangle \cdot \langle 3t^2, -2t, 1\rangle = \int_0^1 (3t^2 \sin t^3 - 2t\cos t^2 + t^4)\, dt$

$= \left[-\cos t^3 - \sin t^2 + \frac{1}{5}t^5\right]_0^1 = \boxed{\frac{6}{5} - \cos 1 - \sin 1}$

(c) $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + xy\vec{k}$, $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$, $0 \le t \le \pi$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \langle \cos t, \sin t, \cos t \sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle\, dt = \int_0^\pi \sin t \cos t\, dt$

$= \frac{1}{2}\sin^2 y\Big]_0^\pi = \boxed{0}$

E.3 The Fundamental Theorem for Line Integrals

Definition. A vector field is **path-independent** or **conservative** if for any two points in the xy-plane P and Q, the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

has the same value no matter which piecewise smooth curve C you take from P to Q.

→ All these curves have the same initial and terminal points.

Theorem (**Fundamental Theorem of Calculus for Line Integrals**)**.** If the gradient vector field for a function f is continuous and the curve C starts at P and ends at Q, then

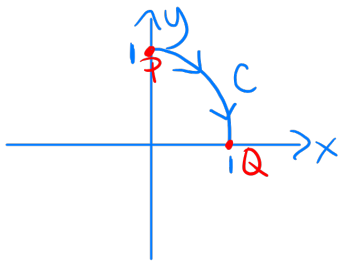
$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$$

* Analogous to the Net Change Theorem from single-variable calculus.

The above theorem implies that **all gradient vector fields are** path-independent (conservative).

The theorem is very useful, since it makes computing flow along (a line integral) of a path-independent vector field very easy.

Example. Let $f(x,y) = \frac{x^2}{2} + xy$ and C be the first quadrant quarter-circle of radius 1, oriented clockwise. Compute $\int_C \nabla f \cdot d\vec{r}$.



$\nabla f(x,y) = \langle x+y, x \rangle$ is continuous

$$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P) = f(1,0) - f(0,1)$$

$$= \frac{1}{2} + \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 0^2 + 0\cdot 1 = \boxed{\frac{1}{2}}$$

In fact, an even stronger statement regarding path-independence and gradient vector fields is true:

Theorem. A continuous vector field $\vec{F}$ defined on an open region is path-independent/conservative *if and only if* $\vec{F}$ is a gradient vector field.

Fact. Path-independent = conservative = gradient vector field

Note. Why would it be very difficult to determine whether a vector field is path-independent using the original definition?

Because we would have to consider all possible curves C between any two points in the vector field and then compute all those integrals.

The above theorem gives an easier way:

Show that $\vec{F} = \nabla f$ for some function f.

Definition. If a vector field $\vec{F}$ is such that $\vec{F} = \nabla f$ for some function f, then f is called the **potential function** for $\vec{F}$.

Example. Show that $\vec{F} = xy^2\vec{i} + x^2y\vec{j}$ is path-independent by finding a potential function.

$\vec{F} = \nabla f = \langle f_x, f_y \rangle \Rightarrow f_x = xy^2$ and $f_y = x^2y$

$f(x,y) = \frac{x^2}{2}\cdot y^2 + g(y) \Rightarrow f_y = \frac{2x^2\cdot y}{2} + g'(y) = x^2y + g'(y)$

$\Rightarrow g'(y) = 0$

$\Rightarrow g(y) = K$ a constant

$f(x,y) = \frac{x^2y^2}{2} + K$

We can verify that $\vec{F} = \nabla f$.

Example. Show that $\vec{F} = (xy^2 + \cos y)\vec{i} + (x^2y - x\sin y)\vec{j}$ is conservative.

Show $\vec{F} = \nabla f$

$\vec{F} = \langle f_x, f_y \rangle \Rightarrow f_x = xy^2 + \cos y$ and $f_y = x^2y - x\sin y$

$f(x,y) = \frac{x^2y^2}{2} + x\cos y + g(y) \Rightarrow f_y = x^2y - x\sin y + g'(y)$

$\Rightarrow g'(y) = 0$

$\Rightarrow g(y) = K$

$f(x,y) = \frac{x^2y^2}{2} + x\cos y + K$

Example. Determine whether or not $\vec{F}(x,y) = (xy+y^2)\vec{i} + (x^2+2xy)\vec{j}$ is a conservative field.

$\vec{F} = \nabla f = \langle f_x, f_y \rangle \Rightarrow f_x = xy+y^2$ and $f_y = x^2+2xy$

$f(x,y) = \frac{x^2y}{2} + xy^2 + g(y) \Rightarrow f_y = \frac{x^2}{2} + 2xy + g'(y)$

$\Rightarrow g(y) = \frac{x^2}{2}\cdot y \Rightarrow f(x,y) = \frac{x^2y}{2} + xy^2 + \frac{x^2}{2}\cdot y = x^2y + xy^2$

$g'(y) = \frac{x^2}{2}$

$\Rightarrow f_x = 2xy + y^2$ $\Rightarrow$ This is a contradiction. $\vec{F}$ is not conservative.

Example. Find a function f such that $\vec{F} = \nabla f$ for $\vec{F}(x,y) = (3+2xy^2)\vec{i} + 2x^2y\vec{j}$. Then use it to evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve C: the arc of the hyperbola $\frac{1}{x}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

$\vec{F} = \nabla f = \langle f_x, f_y \rangle \Rightarrow f_x = 3+2xy^2$ and $f_y = 2x^2y$

$\Rightarrow f(x,y) = 3x + x^2y^2 + g(y) \Rightarrow f_y = 0 + 2x^2y + g'(y)$

$\Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, a constant

$\Rightarrow f(x,y) = 3x + x^2y^2 + K$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f\left(4, \frac{1}{4}\right) - f(1,1)$

$= 3\cdot 4 + 4^2\cdot\left(\frac{1}{4}\right)^2 + K - [3+1+K]$

$= 12+1-4 = \boxed{9}$

Homework E.3 - The Fundamental Theorem for Line Integrals

Prof. Leyda Almodóvar

Name: _____

1. Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$.

   (a) $\vec{F}(x,y) = (y^2 - 2x)\vec{i} + 2xy\vec{j}$

   If $\vec{F} = \nabla f$ then $f_x(x,y) = y^2 - 2x$ and $f_y(x,y) = 2xy$. Then $f_x(x,y) = y^2 - 2x$ implies $f(x,y) = xy^2 - x^2 + g(y)$ and $f_y(x,y) = 2xy + g'(y)$. This implies $g'(y) = 0$ and $g(y) = K$ for a constant K. Hence, $f(x,y) = xy^2 - x^2 + K$ is a potential function for $\vec{F}$, which implies $\vec{F}$ is a conservative vector field.

   (b) $\vec{F}(x,y) = y^2 e^{xy}\vec{i} + (1+xy)e^{xy}\vec{j}$

   If $\vec{F} = \nabla f$ then $f_x(x,y) = y^2 e^{xy}$ and $f_y(x,y) = (1+xy)e^{xy}$. Then $f(x,y) = ye^{xy} + g(y)$ and $f_y(x,y) = e^{xy} + xye^{xy} + g'(y)$. Hence, $g'(y) = 0$ and $g(y) = K$ for a constant K. Thus, $f(x,y) = ye^{xy} + K$ is a potential function for $\vec{F}$, which implies $\vec{F}$ is a conservative vector field.

   (c) $\vec{F}(x,y) = ye^x\vec{i} + (e^x + e^y)\vec{j}$

   If $\vec{F} = \nabla f$ then $f_x(x,y) = ye^x$ and $f_y(x,y) = e^x + e^y$. Then $f(x,y) = ye^x + g(y)$ and $f_y(x,y) = e^x + g'(y)$. Hence, $g'(y) = e^y$ and $g(y) = e^y$. Thus, $f(x,y) = ye^x + e^y$ is a potential function for $\vec{F}$, which implies $\vec{F}$ is a conservative vector field.

   (d) $\vec{F}(x,y) = (ye^x + \sin y)\vec{i} + (e^x + x\cos y)\vec{j}$

   If $\vec{F} = \nabla f$ then $f_x(x,y) = ye^x + \sin y$ and $f_y(x,y) = e^x + x\cos y$. Then $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. Hence, $g'(y) = 0$ and $g(y) = K$ for a constant K. Thus, $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\vec{F}$, which implies $\vec{F}$ is a conservative vector field.

2. Find a function f such that $\vec{F} = \nabla f$ and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the given curve C.

(a) $\vec{F}(x,y) = x^2y^3\vec{i} + x^3y^2\vec{j}$, $C : \vec{r}(t) = \langle t^3 - 2t, t^3 + 2t\rangle$, $0 \le t \le 1$

If $\vec{F} = \nabla f$ then $f_x(x,y) = x^2y^3$ and $f_y(x,y) = x^3y^2$. Then $f_x(x,y) = x^2y^3$ implies $f(x,y) = \frac{x^3}{3}y^3 + g(y)$ and $f_y(x,y) = x^3y^2 + g'(y)$. Hence, $g'(y) = 0$ and $g(y) = K$ for a constant K. We can take $K = 0$, thus, $f(x,y) = \frac{1}{3}x^3y^3$.

Note that if $t = 0$, $x = 0$ and $y = 0$ and if $t = 1$, $x = -1$ and $y = 3$. Then,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(-1,3) - f(0,0) = -9 - 0 = \boxed{-9}$$

(b) $\vec{F}(x,y) = (1+xy)e^{xy}\vec{i} + x^2e^{xy}\vec{j}$, $C : \vec{r}(t) = \cos t\vec{i} + 2\sin t\vec{j}$, $0 \le t \le \frac{\pi}{2}$

If $\vec{F} = \nabla f$ then $f_x(x,y) = (1+xy)e^{xy}$ and $f_y(x,y) = x^2e^{xy}$. Note that it's easier to integrate f_y than f_x. Then $f_y(x,y) = x^2e^{xy}$ implies $f(x,y) = xe^{xy} + g(x)$ and $f_x(x,y) = e^{xy} + xye^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. Hence, $g'(x) = 0$ and $g(x) = K$ for a constant K. We can take $K = 0$, thus $f(x,y) = xe^{xy}$.

Note that if $t = 0$, $x = 1$ and $y = 0$ and if $t = \frac{\pi}{2}$, $x = 0$ and $y = 2$. Then,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(0,2) - f(1,0) = 0 - e^0 = \boxed{-1}$$

E.4 Green's Theorem and Curl

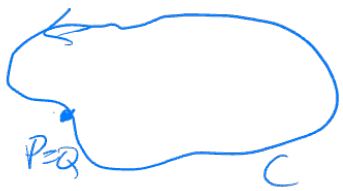
Recall that $\vec{F}$ is **path-indpendent** or **conservative** if for any two points in the xy-plane P and Q, the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

has the same value no matter which piecewise smooth curve C you take from P to Q.

So, what does this mean if C is a *closed* curve ($P = Q$)?

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(Q) - f(P)$
$= 0$



Theorem. A vector field is path-independent if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.

- Recall that the line integral of $\vec{F}$ over C represents the "flow along" or circulation of $\vec{F}$ along C. So **a vector field is path-independent if and only if it has zero circulation on closed curves.**

We present another way to figure out whether or not a vector field is gradient.

Definition. For a vector field $\vec{F} = F_1\vec{i} + F_2\vec{j}$, we define the **curl** of $\vec{F}$ as

$$\text{Curl}\,\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

- The curl's value is a measure of how much the vectors "turn" or "curl" about a point (x, y).
- If $\vec{F}$ is a gradient vector field, then the curl of $\vec{F}$ is equal to 0.
- If the curl of $\vec{F}$ is not zero, $\vec{F}$ cannot be gradient.

Example. Find the curl of $\vec{F} = \langle 2x^2 + y, 4xy \rangle$.

$F_1 = 2x^2 + y, \quad F_2 = 4xy$

$\text{Curl}\,\vec{F} = \frac{\partial}{\partial x}(4xy) - \frac{\partial}{\partial y}(2x^2 + y) = \boxed{4y - 1}$

Note $\vec{F}$ is not conservative since $\text{curl}\,\vec{F} \neq 0$.

Curl in Three-Dimensions

Definition. For a vector field $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, we define the **curl** of $\vec{F}$ as

$$\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Note. We introduce the vector differential operator ∇ "del" as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \qquad \nabla f = \langle f_x, f_y, f_z \rangle$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} = \text{curl } \vec{F}$$

- In two-dimensions, the curl is a scalar. In three-dimensions, the curl is a vector field.
- If $\vec{F}$ is a three-dimensional gradient vector field, then $\text{curl}(\vec{F}) = \vec{0}$.
- The reason for the name *curl* is that the curl vector is associated with rotations. For example, suppose $\vec{F}$ represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of $\text{curl}(\vec{F})$, and $|\text{curl}(\vec{F})|$ measures how quickly the particles move around the axis.

Example. If $\vec{F}(x, y, z) = xy^2z^2\vec{i} + x^2yz^2\vec{j} + x^2y^2z\vec{k}$, find $\text{curl}(\vec{F})$.

$$\text{Curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = \left(\frac{\partial}{\partial y}(x^2y^2z) - \frac{\partial}{\partial z}(x^2yz^2)\right)\vec{i} - \left(\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial z}(xy^2z^2)\right)\vec{j} + \left(\frac{\partial}{\partial x}(x^2yz^2) - \frac{\partial}{\partial y}(xy^2z^2)\right)\vec{k}$$

$$= (2x^2yz - 2x^2yz)\vec{i} - (2xy^2z - 2xy^2z)\vec{j} + (2xyz^2 - 2xyz^2)\vec{k}$$

$$= \vec{0}$$

$\ast \vec{F}$ is conservative/gradient

Example. Show that the vector field $\vec{F}(x, y, z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$ is not conservative.

$$\text{Curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\vec{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\vec{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\vec{k}$$

$$= (-2y - xy)\vec{i} - (0 - x)\vec{j} + (yz - 0)\vec{k}$$

$$= (-2y - xy)\vec{i} + x\vec{j} + yz\vec{k} \neq \vec{0}$$

$\Rightarrow \vec{F}$ is not conservative.

Example. Show that the vector field $\vec{F}(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k}$ is conservative. Then, find a function f such that $\vec{F} = \nabla f$.

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\vec{i} - (3y^2z^2 - 3y^2z^2)\vec{j} + (2yz^3 - 2yz^3)\vec{k} = \vec{0} \Rightarrow \vec{F} = \nabla f$$

$f_x = y^2z^3 \Rightarrow f(x,y,z) = xy^2z^3 + g(y,z) = xy^2z^3 + h(z)$

$f_y = 2xyz^3 \quad f_y = 2xyz^3 + g_y(y,z) \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$

$f_z = 3xy^2z^2 \quad f_z = 3xy^2z^2 + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = K$, a constant

$$\boxed{f(x,y,z) = xy^2z^3 + K}$$

Definition. For a vector field $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, we define the **divergence** of $\vec{F}$ as

$$\boxed{\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

- div $\vec{F} = \nabla \cdot \vec{F}$
- curl $\vec{F}$ is a vector field but div $\vec{F}$ is a scalar field. (a function)

Note. The reason for the name *divergence* can be understood in the context of fluid flow. If $\vec{F}$ is the velocity of a fluid then div $\vec{F}$ measures the tendency of the fluid to diverge from the point (x,y,z).

Example. If $\vec{F}(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$, find div $\vec{F}$.

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz + 0 = \boxed{z(1+x)}$$

Fact. div curl $\vec{F} = 0$ $\Rightarrow$ If div $\vec{G} \neq 0$, then $\vec{G} \neq \text{curl}\,\vec{F}$

Example. Verify the fact stated above on the vector field $\vec{F}(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$.

We found $\text{curl}\vec{F} = (-2y - xy)\vec{i} + x\vec{j} + yz\vec{k}$

$$\text{div}(\text{curl}\vec{F}) = \frac{\partial}{\partial x}(-2y - xy) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(yz)$$

$$= -y + 0 + y = \boxed{0}$$

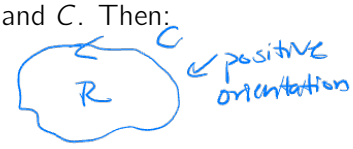
Green's Theorem

→ does not cross itself
→ initial point = terminal point

Theorem (Green's Theorem). Let C be a piecewise smooth simple closed curve that is the boundary of a region R in the xy-plane oriented so that the region is on the left as we move along the curve. Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a smooth vector field on an open region containing R and C. Then:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

curl $\vec{F}$ in 2D

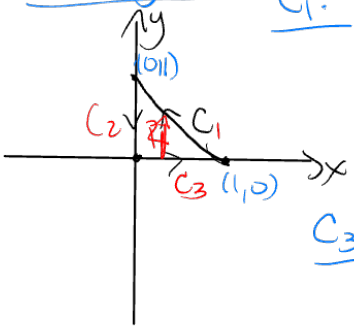


Note. Green's Theorem allows us to switch from a line integral along a curve to a simple double integral over the enclosed region.

Notation: $\oint_C F_1\,dx + F_2\,dy$ ← C has positive orientation

Example. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segment from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Line integral:



C_1: $x(t) = (1-t)(1) + t\cdot 0 = 1-t$, $y(t) = (1-t)\cdot 0 + t\cdot 1 = t$, $0 \le t \le 1$, $dx = -dt$, $dy = dt$

C_2: $x(t) = (1-t)\cdot 0 + t\cdot 0 = 0$, $y(t) = (1-t)\cdot 0 + t\cdot 1 = t$, $0 \le t \le 1$, $dy = dt$, $dx = 0\,dt$

Note! $\int_{C_2} x^4\,dx + xy\,dy = 0$

C_3: $x(t) = (1-t)\cdot 0 + t\cdot 1 = t$, $y(t) = (1-t)\cdot 0 + t\cdot 0 = 0$, $0 \le t \le 1$, $dx = dt$, $dy = 0\,dt$

$$\int_C x^4\,dx + xy\,dy = \underbrace{\int_0^1 (1-t)^4(-dt) + \int_0^1 (1-t)\,t\,dt}_{C_1} + \underbrace{\int_0^1 t^4\,dt}_{C_3} = -\int_0^1 (1-t)^4\,dt + \int_0^1 t - t^2\,dt + \int_0^1 t^4\,dt$$

($u = 1-t$, $du = -dt$)

$$= \int_1^0 u^4\,du + \left[\frac{t^2}{2} - \frac{t^3}{3} + \frac{t^5}{5}\right]_0^1 = \frac{u^5}{5}\Big]_1^0 + \left[\frac{1}{2} - \frac{1}{3} + \frac{1}{5}\right] = -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + \frac{1}{5} = \boxed{\frac{1}{6}}$$

Green's Theorem: $\int_C x^4\,dx + xy\,dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$

$F_1 = x^4$, $F_2 = xy$

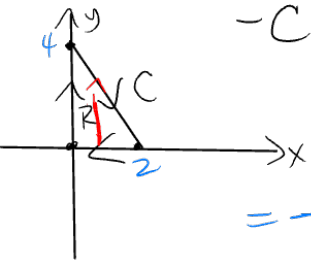
$R = \{(x,y) \mid 0 \le y \le 1-x,\ 0 \le x \le 1\}$ described as Type I

$$= \int_0^1 \int_0^{1-x} y - 0\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2}\,dx$$

($u = 1-x$, $du = -dx$)

$$= \int_1^0 \frac{-u^2}{2}\,du = -\frac{u^3}{6}\Big]_1^0 = 0 + \frac{1}{6} = \boxed{1/6}$$

Example. Use Green's Theorem to evaluate $\int_C \vec{F}\,d\vec{r}$ where $\vec{F} = (y\cos x - xy\sin x)\vec{i} + (xy + x\cos x)\vec{j}$ and C is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.



$-C$ gives the positive orientation

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} (y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy$$

$$= -\iint \frac{\partial}{\partial x}(xy + x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\sin x)\,dA$$

$$= -\iint y + \cos x - x\sin x - \cos x + x\sin x\,dA = -\int_0^2 \int_0^{4-2x} y\,dy\,dx$$

$R = \{(x,y) \mid 0 \le y \le 4-2x,\ 0 \le x \le 2\}$

$$= -\int_0^2 \left[\frac{y^2}{2}\right]_0^{4-2x} dx = -\int_0^2 \frac{(4-2x)^2}{2}\,dx$$

$$= \int_0^2 \frac{16 - 16x + 4x^2}{2}\,dx = \int_0^2 8 - 8x + 2x^2\,dx = \left[8x - 4x^2 + 2\frac{x^3}{3}\right]_0^2$$

$$= \left[16 - 16 + \frac{16}{3} - (0 - 0 + 0)\right] = \boxed{\frac{16}{3}}$$

Note. Recall what the line integral represents: the flow along the curve C / circulation on the curve C of the vector field $\vec{F}$. Green's Theorem compels us to integrate over the whole region, which gives us the circulation inside the region (imagine a stick in a pool of water in the shape of R, how would it spin?) What does this mean?

Fact. Green's Theorem tells us that the sum of the circulation / microscopic spin on the boundary is equal to the sum of the circulation inside the region.

Relation Between Curl, Divergence, and Green's Theorem

Suppose that the plane region R, its boundary curve C, and the functions F_1 and F_2 satisfy the hypotheses of Green's Theorem. Then for the vector field $\vec{F} = F_1\vec{i} + F_2\vec{j}$, we obtain

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int\int_R (\text{curl } \vec{F})\cdot\vec{k}\,dA}$$

If the curve C is given by $r(t) = x(t)\vec{i} + y(t)\vec{j}$ and $\vec{n}$ is the outward unit normal vector to C, then

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds = \int\int_R \text{div } \vec{F}(x,y)\,dA}$$

Homework E.4 - Green's Theorem and Curl

Prof. Leyda Almodóvar

Name: _____

1. Find the curl and the divergence of the vector field.

   (a) $\vec{F}(x,y,z) = x^3yz^2\vec{j} + y^4z^3\vec{k}$

   $$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix}$$

   $$= \left(\frac{\partial(y^4z^3)}{\partial y} - \frac{\partial(x^3yz^2)}{\partial z}\right)\vec{i} - \left(\frac{\partial(y^4z^3)}{\partial x} - \frac{\partial(0)}{\partial z}\right)\vec{j} + \left(\frac{\partial(x^3yz^2)}{\partial x} - \frac{\partial(0)}{\partial y}\right)\vec{k}$$

   $$= (4y^3z^3 - 2x^3yz)\vec{i} - (0-0)\vec{j} + (3x^2yz^2 - 0)\vec{k} = \boxed{(4y^3z^3 - 2x^3yz)\vec{i} + 3x^2yz^2\vec{k}}$$

   $$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(y^4z^3) = \boxed{x^3z^2 + 3y^4z^2}$$

   (b) $\vec{F}(x,y,z) = xye^z\vec{i} + yze^x\vec{k}$

   $$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xye^z & 0 & yze^x \end{vmatrix}$$

   $$= \left(\frac{\partial(yze^x)}{\partial y} - \frac{\partial(0)}{\partial z}\right)\vec{i} - \left(\frac{\partial(yze^x)}{\partial x} - \frac{\partial(xye^z)}{\partial z}\right)\vec{j} + \left(\frac{\partial(0)}{\partial x} - \frac{\partial(xye^z)}{\partial y}\right)\vec{k}$$

   $$= \boxed{ze^x\vec{i} + (xye^z - yze^x)\vec{j} - xe^z\vec{k}}$$

   $$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(yze^x) = \boxed{ye^z + ye^x}$$

   (c) $\vec{F}(x,y,z) = \sin yz\vec{i} + \sin zx\vec{j} + \sin xy\vec{k}$

   $$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin yz & \sin zx & \sin xy \end{vmatrix}$$

$$= \left(\frac{\partial(\sin xy)}{\partial y} - \frac{\partial(\sin zx)}{\partial z}\right)\vec{i} - \left(\frac{\partial(\sin xy)}{\partial x} - \frac{\partial(\sin yz)}{\partial z}\right)\vec{j} + \left(\frac{\partial(\sin zx)}{\partial x} - \frac{\partial(\sin yz)}{\partial y}\right)\vec{k}$$

$$= \boxed{(x\cos xy - x\cos zx)\vec{i} - (y\cos xy - y\cos yz)\vec{j} + (z\cos zx - z\cos yz)\vec{k}}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(\sin yz) + \frac{\partial}{\partial y}(\sin zx) + \frac{\partial}{\partial z}(\sin xy) = 0 + 0 + 0 = \boxed{0}$$

2. Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\vec{F} = \nabla f$.

(a) $\vec{F}(x, y, z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \left(\frac{\partial(3xy^2z^2)}{\partial y} - \frac{\partial(2xyz^3)}{\partial z}\right)\vec{i} - \left(\frac{\partial(3xy^2z^2)}{\partial x} - \frac{\partial(y^2z^3)}{\partial z}\right)\vec{j} + \left(\frac{\partial(2xyz^3)}{\partial x} - \frac{\partial(y^2z^3)}{\partial y}\right)\vec{k}$$

$$= (6xyz^2 - 6xyz^2)\vec{i} + (3y^2z^2 - 3y^2z^2)\vec{j} + (2yz^3 - 2yz^3)\vec{k} = \vec{0}$$

Hence, $\vec{F}$ is conservative and $\vec{F} = \nabla f$ for some potential function f. Then $f_x(x, y, z) = y^2z^3$ implies $f(x, y, z) = xy^2z^3 + g(y, z)$

$\Rightarrow f_y(x, y, z) = 2xyz^3 + g_y(y, z)$. But, $f_y = 2xyz^3$ so $g(y, z) = h(z)$ and $f(x, y, z) = xy^2z^3 + h(z)$.

Thus, $f_z(x, y, z) = 3xy^2z^2 + h'(z)$ but $f_z = 3xy^2z^2$ so $h(z) = K$, a constant. Hence, a potential function for $\vec{F}$ is $f(x, y, z) = xy^2z^3 + K$.

(b) $\vec{F}(x, y, z) = xyz^4\vec{i} + x^3z^4\vec{j} + 4x^2yz^3\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^4 & x^3z^4 & 4x^2yz^3 \end{vmatrix}$$

$= (64x^2z^3 - 4x^2z^3)\vec{i} - (8xyz^3 - 4xyz^3)\vec{j} + (2xz^4 - xz^4)\vec{k} \neq \vec{0}$, so $\vec{F}$ is not conservative.

(c) $\vec{F}(x, y, z) = z\cos y\vec{i} + xz\sin y\vec{j} + x\cos y\vec{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\cos y & xz\sin y & x\cos y \end{vmatrix}$$

$$= (-x\sin y - x\sin y)\vec{i} - (\cos y - \cos y)\vec{j} + [z\sin y - (-z\sin y)]\vec{k}$$

$= -2x\sin y\vec{i} + 2z\sin y\vec{k} \neq \vec{0}$, so $\vec{F}$ is not conservative.

3. Is there a vector field $\vec{F}$ such that curl $\vec{F} = \langle x\sin y, \cos y, z - xy\rangle$? Explain.

Suppose there exists $\vec{F}$ such that curl $\vec{F} = \langle x\sin y, \cos y, z - xy\rangle$. Then div(curl $\vec{F}$) $= 0$. But, div (curl $\vec{F}$) $= \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial z}(z - xy) = \sin y - \sin y + 1 \neq 0$.

4. Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

(a) $\int_C y\,dx - x\,dy$, where C is the circle with center the origin and radius 4.

(a) Parametric equations for C are $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. Then $dx = -4\sin t\,dt$, $dy = 4\cos t\,dt$ and

$\oint_C y\,dx - x\,dy = \int_0^{2\pi}[(4\sin t)(-4\sin t) - (4\cos t)(4\cos t)]\,dt$

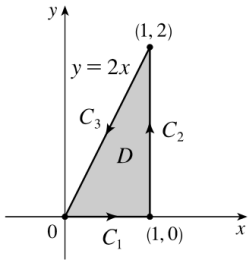
$= -16\int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = -16\int_0^{2\pi} 1\,dt = \boxed{-32\pi}$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,

$\oint_C y\,dx - x\,dy = \int\int_R\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right]dA = \int\int_R(-1-1)\,dA = -2\int\int_R dA$

$= -2(\text{area of } R) = -2\cdot\pi(4)^2 = \boxed{-32\pi}$

(b) $\int_C xy\,dx + x^2y^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.



(a) On C_1: $x = t$, $y = 0$, $\Rightarrow$ $dx = dt$ and $dy = 0\,dt$, $0 \le t \le 1$.
On C_2: $x = 1$, $y = 2t$, $\Rightarrow$ $dx = 0\,dt$ and $dy = 2\,dt$, $0 \le t \le 1$.
On C_3: $x = 1 - t$, $y = 2 - 2t$, $\Rightarrow$ $dx = -\,dt$, $dy = -2\,dt$, $0 \le t \le 1$.

$\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$

$= \int_0^1 0\,dt + \int_0^1 16t^3\,dt + \int_0^1[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3]\,dt$

$= 0 + \left[4t^4\right]_0^1 + \int_0^1 [-2(1-t)^2 - 16(1-t)^5]\,dt$

$= 4 + \left[\frac{2}{3}(1-t)^3 + \frac{8}{3}(1-t)^6\right]_0^1 = 4 + 0 - \frac{10}{3} = \boxed{\frac{2}{3}}$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \int\int_R \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$= \int_0^1 \left[\frac{1}{2}xy^4 - xy\right]_0^{2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \frac{4}{3} - \frac{2}{3} = \boxed{\frac{2}{3}}$

5. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

   (a) $\int_C ye^x\,dx + 2e^x\,dy$, C is the rectangle with vertices $(0,0)$, $(3,0)$, $(3,4)$, and $(0,4)$.

   The region R enclosed by C is $[0,3] \times [0,4]$, so

   $\oint_C ye^x\,dx + 2e^x\,dy = \int\int_R \left[\frac{\partial}{\partial x}(2e^x) - \frac{\partial}{\partial y}(ye^x)\right] dA = \int_0^3 \int_0^4 (2e^x - e^x)\,dy\,dx$

   $= \int_0^3 e^x\,dx \int_0^4 dy = [e^x]_0^3[y]_0^4 = (e^3 - e^0)(4-0) = \boxed{4(e^3-1)}$

   (b) $\int_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

   $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy = \int\int_R \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})\right] dA$

   $= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)\,dy\,dx = \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \boxed{\frac{1}{3}}$

   (c) $\int_C y^3\,dx - x^3\,dy$, where C is the circle $x^2 + y^2 = 4$.

   $\oint_C y^3\,dx - x^3\,dy = \int\int_R \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial}(y^3)\right] dA = \int\int_R (-3x^2 - 3y^2)\,dA$

   $= \int_0^{2\pi} \int_0^2 (-3r^2) r\,dr\,d\theta = -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3\left[\theta\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^2$

   $= -3(2\pi)(4) = \boxed{-24\pi}$

6. Use Green Theorem's to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle y - \cos y, x \sin y\rangle$, and C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.

   The region R enclosed by C is the disk with radius 2 centered at $(3,-4)$. Note that C is traversed clockwise, so $-C$ gives the positive orientation. Then,

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} (y - \cos y)\, dx + (x \sin y)\, dy = -\int\int_R \left[\frac{\partial}{\partial x}(x \sin y) - \frac{\partial}{\partial y}(y - \cos y) \right] dA$$

$$= -\int\int_R (\sin y - 1 - \sin y)\, dA = \int\int_R dA = \text{ area of } R = \pi(2)^2 = \boxed{4\pi}$$

E.5 Parametric Surfaces and Their Areas

In this section we use vector functions to describe general surfaces, called *parametric surfaces*, and compute their areas.

Recall that we parameterized curves by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j}$ where $x = g(t)$ and $y = h(t)$ and $a \le t \le b$. We can take a similar approach by describing a surface by a vector function $\vec{r}(u, v)$ of two parameters u and v.

Definition. Let S be a surface and (u, v) be points in a two-dimensional space D, then

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

is called the **parametric representation** of the **parametric surface** S. The equations

$$x = x(u, v), y = y(u, v), z = z(u, v)$$

are called **parametric equations** of S.

- Each choice of u and v gives a point on S; by making all choices, we get all of S.
- S is traced out by the tip of the vector $\vec{r}(u, v)$ as (u, v) moves throughout the region D.
- Parametric representations of curves are not unique.

Example. Identify the surface with vector equation $\vec{r}(u, v) = u^2\vec{i} + u\cos v\vec{j} + u\sin v\vec{k}$.

The parametric equations are $x = u^2$, $y = u\cos v$, $z = u\sin v$

We notice y and z involve $\cos v$ and $\sin v$, respectively.

$y^2 = u^2\cos^2 v$, $z^2 = u^2\sin^2 v \Rightarrow y^2 + z^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = x$

$\Rightarrow$ The surface is $x = y^2 + z^2$, which we recognize as a circular paraboloid whose axis is the x-axis.

Example. Find a parametric representation for the elliptic paraboloid $y = 5x^2 + 4z^2$.

Notice $y = f(x, z)$. Let x and z be the parameters.

$x = x$, $z = z$, $y = 5x^2 + 4z^2$

$$\vec{r}(x, z) = x\vec{i} + (5x^2 + 4z^2)\vec{j} + z\vec{k}$$

- If a surface if given in the form $z = f(x, y)$, then $\vec{r}(x, y) = x\vec{i} + y\vec{j} + f(x, y)\vec{k}$.
- If a surface if given in the form $x = f(y, z)$, then $\vec{r}(y, z) = f(y, z)\vec{i} + y\vec{j} + z\vec{k}$.
- If a surface if given in the form $y = f(x, z)$, then $\vec{r}(x, z) = x\vec{i} + f(x, z)\vec{j} + z\vec{k}$.

Example. Find a parametric representation for the (1 part of the) hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane. $\Rightarrow x \geq 0$

Since we want the portion that corresponds to $x \geq 0$, we solve the equation for x:

$4x^2 = 4 + 4y^2 + z^2 \Rightarrow x^2 = 1 + y^2 + \frac{z^2}{4} \Rightarrow x = \sqrt{1 + y^2 + \frac{z^2}{4}} = f(y, z)$

Parametric equations: $y = y$, $z = z$, $x = \sqrt{1 + y^2 + \frac{z^2}{4}}$

$$\vec{r}(y, z) = \sqrt{1 + y^2 + \frac{z^2}{4}}\,\vec{i} + y\vec{j} + z\vec{k}$$

Note: Sometimes parameters have restrictions

Example. Find a parametric representation for the sphere $x^2 + y^2 + z^2 = 25$.

Spheres have a simple representation in spherical coordinates:

$\rho^2 = 25 \Rightarrow \rho = 5$

We choose the angles θ and ϕ as parameters:

$x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$, $z = 5\cos\phi$

$$\vec{r}(\phi, \theta) = 5\sin\phi\cos\theta\,\vec{i} + 5\sin\phi\sin\theta\,\vec{j} + 5\cos\phi\,\vec{k}$$

where $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$

Tangent Planes

We find the tangent plane to a parametric surface S traced out by a vector function

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}$$

at a point P_0 with position vector $\vec{r}(u_0, v_0)$.

If we keep $u = u_0$ constant, then $\vec{r}(u_0, v)$ becomes a vector function of a single parameter v and defines a curve C_1 lying on S. Then the tangent vector to C_1 at P_0 is given by

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0, v_0)\vec{k}$$

Similarly, if $v = v_0$ is constant, then $\vec{r}(u, v_0)$ defines a curve C_2 lying on S. Then the tangent vector to C_2 at P_0 is given by

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0, v_0)\vec{k}$$

$\vec{r}_u \times \vec{r}_v = -\vec{r}_v \times \vec{r}_u$

If $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, the vector $\vec{r}_u \times \vec{r}_v$ will be orthogonal to S so it can be used as the normal vector needed to write down the equation of a tangent plane.

Recall: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ where $\langle a, b, c \rangle$ is normal to S and $P_0 = (x_0, y_0, z_0)$

Example. Find an equation of the tangent plane to the surface given by parametric equations $x = u+v$, $y = 3u^2$, $z = u - v$ at the point $(2, 3, 0)$.

$\vec{r}(u,v) = (u+v)\vec{i} + 3u^2\vec{j} + (u-v)\vec{k}$

$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} = \vec{i} + 6u\vec{j} + \vec{k}$

$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k} = \vec{i} + 0\vec{j} - \vec{k} = \vec{i} - \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6u & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 6u & 1 \\ 0 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 6u \\ 1 & 0 \end{vmatrix}\vec{k}$$

$$= -6u\vec{i} - (-2)\vec{j} - 6u\vec{k} = \langle -6u, 2, -6u \rangle$$

To find the values of u and v corresponding to $(2,3,0)$:

$2 = u+v$, $3 = 3u^2$, $0 = u - v$ $\Rightarrow u = v$, $\Rightarrow 2u = 2 \Rightarrow u = 1$

($3 = 3u^2 \Rightarrow u^2 = 1 \Rightarrow u = \pm 1$)

$\Rightarrow \boxed{u = v = 1}$

A normal vector to the surface at $(2,3,0)$ is $\langle -6, 2, -6 \rangle$

and an equation of the tangent plane is

$$-6(x-2) + 2(y-3) - 6(z-0) = 0$$

$$-6x + 12 + 2y - 6 - 6z = 0$$

$$\boxed{-6x + 2y - 6z = -6}$$

or

$$-3x + y - 3z = -3$$

Example. Find an equation of the tangent plane to the surface given by $\vec{r}(u, v) = u\cos v\vec{i} + u\sin v\vec{j} + v\vec{k}$ at the point given by $u = 1$, $v = \frac{\pi}{3}$.

$x = u\cos v$, $y = u\sin v$, $z = v$

$x = 1\cdot\cos\pi/3 = 1/2$, $y = 1\cdot\sin\pi/3 = \frac{\sqrt{3}}{2}$, $z = \pi/3$

Point is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \pi/3\right)$

$\vec{r}_u = \cos v\,\vec{i} + \sin v\,\vec{j} + 0\vec{k} = \cos v\,\vec{i} + \sin v\,\vec{j}$

$\vec{r}_v = -u\sin v\,\vec{i} + u\cos v\,\vec{j} + \vec{k}$

$\vec{r}_u(1, \pi/3) = \cos\pi/3\,\vec{i} + \sin\pi/3\,\vec{j} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$

$\vec{r}_v(1, \pi/3) = -\sin\pi/3\,\vec{i} + \cos\pi/3\,\vec{j} + k = -\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}$

$$\vec{r}_u(1,\pi/3) \times \vec{r}_v(1,\pi/3) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 1/2 & 1 \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{3}}{2} & 0 \\ 1/2 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 1/2 \end{vmatrix}\vec{k}$$

$$= \frac{\sqrt{3}}{2}\vec{i} - \frac{1}{2}\vec{j} + \left(\frac{1}{4} + \frac{3}{4}\right)\vec{k} = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \right\rangle$$

An equation of the tangent plane at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \pi/3\right)$ is

$$\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + 1\cdot\left(z - \pi/3\right) = 0$$

$$\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{4} - \frac{1}{2}y + \frac{\sqrt{3}}{4} + z - \pi/3 = 0$$

$$\boxed{\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \pi/3}$$

Surface Area

Definition. If the parametric surface S is traced out exactly once as (u, v) ranges over the points on D, the **surface area** of S is given by

$$A = \int\int_D |\vec{r}_u \times \vec{r}_v| \, dA$$

The order doesn't matter! $|\vec{r}_u \times \vec{r}_v| = |\vec{r}_v \times \vec{r}_u|$ ← magnitude!

$\vec{r}(u,v)$ is the parametric representation of S.

Example. Find the area of the part of the plane with parametric equation $\vec{r}(u, v) = (u + v)\vec{i} + (2 - 3u)\vec{j} + (1 + u - v)\vec{k}$ that is given by $0 \le u \le 2$, $-1 \le v \le 1$.

$\vec{r}_u = \vec{i} - 3\vec{j} + \vec{k}$

$\vec{r}_v = \vec{i} + 0\vec{j} - \vec{k} = \vec{i} - \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 0 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -3 \\ 1 & 0 \end{vmatrix}\vec{k}$$

$$= 3\vec{i} + 2\vec{j} + 3\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{3^2 + 2^2 + 3^2} = \sqrt{22}$$

$D = \{(u,v) \mid 0 \le u \le 2, \ -1 \le v \le 1\}$

$$A = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \iint_D \sqrt{22} \, dA = \int_0^2 \int_{-1}^1 \sqrt{22} \, dv \, du$$

$$= \sqrt{22} \cdot \int_0^2 dv \cdot \int_{-1}^1 du = \sqrt{22}(2-0)(1+1) = \boxed{4\sqrt{22}}$$

Example. Find the area of the surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}})$, $0 \le x \le 1$, $0 \le y \le 1$.

Since $z = f(x,y)$, $x = x$, $y = y$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \frac{2}{3}(x^{3/2} + y^{3/2})\vec{k}$$

$$\vec{r}_x = \vec{i} + 0\vec{j} + \frac{2}{3}\cdot\frac{3}{2}x^{1/2}\vec{k} = \vec{i} + x^{1/2}\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} + \frac{2}{3}\cdot\frac{3}{2}y^{1/2}\vec{k} = \vec{j} + y^{1/2}\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & x^{1/2} \\ 0 & 1 & y^{1/2} \end{vmatrix} = \begin{vmatrix} 0 & x^{1/2} \\ 1 & y^{1/2} \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & x^{1/2} \\ 0 & y^{1/2} \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{k}$$

$$= -x^{1/2}\vec{i} - y^{1/2}\vec{j} + \vec{k} = \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(-x^{1/2})^2 + (-y^{1/2})^2 + 1^2} = \sqrt{x + y + 1}$$

$D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$

$$A = \iint_D \sqrt{x+y+1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dy\, dx$$

$u = x + y + 1$
$du = dy$
$y = 0$, $u = x + 1$
$y = 1$, $u = x + 2$

$$= \int_0^1 \int_{x+1}^{x+2} \sqrt{u}\, du\, dx = \int_0^1 \frac{2}{3}u^{3/2}\Big]_{x+1}^{x+2} dx$$

$$= \int_0^1 \frac{2}{3}(x+2)^{3/2} - \frac{2}{3}(x+1)^{3/2} dx = \frac{2}{3}\left[\frac{2}{5}(x+2)^{5/2} - \frac{2}{5}(x+1)^{5/2}\right]_0^1$$

$$= \frac{4}{15}\left[3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right] = \boxed{\frac{4}{15}\left[3^{5/2} - 2^{7/2} + 1\right]}$$

Homework E.5 - Parametric Surfaces and Their Areas

Prof. Leyda Almodóvar

Name: _____

1. Identify the surface with parametric representation $\vec{r}(u,v) = u\cos v\vec{i} + u\sin v\vec{j} + u\vec{k}$.

   The parametric equations are $x = u\cos v$, $y = u\sin v$, $z = u$. For any point (x,y,z) on the surface we have $x^2 + y^2 = u^2\cos^2 v + u^2\sin^2 v = u^2 = z^2$. We recognize the surface $z^2 = x^2 + y^2$ as a circular cone with axis the z-axis.

2. Find a parametric representation for the surface.

   (a) The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane.

   Solving the equation for y gives $y^2 = \frac{1}{2}(1 - x^2 - 3z^2) \Rightarrow y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$ (since we want the part of the ellipsoid that corresponds to $y \leq 0$). If we let x and z be the parameters, then the parametric equations are $x = x$, $z = z$, $y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$. Hence, $\boxed{\vec{r}(x,z) = x\vec{i} - \sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}\vec{j} + z\vec{k}}$

   (b) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

   Solution 1:
   Note that $z^2 = x^2 + y^2$. Hence, the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$. Since we want the portion of the sphere above this intersection, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \leq 2$. Hence, $\boxed{\vec{r}(x,y) = x\vec{i} + y\vec{j} + \sqrt{4 - x^2 - y^2}\vec{k}}$

   Solution 2:
   Using spherical coordinates, we notice that $\rho = 2$ and parametric equations are $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq \theta \leq 2\pi$. Hence, $\boxed{\vec{r}(\phi,\theta) = 2\sin\phi\cos\theta\vec{i} + 2\sin\phi\sin\theta\vec{j} + 2\cos\phi\vec{k}}$

3. Find an equation of the tangent plane to the given parametric surface at the specified point.

(a) $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$; $(5, 2, 3)$

$\vec{r}(u, v) = (u^2+1)\vec{i} + (v^3+1)\vec{j} + (u+v)\vec{k}$. Then, $\vec{r}_u = 2u\vec{i} + \vec{k}$ and $\vec{r}_v = 3v^2\vec{j} + \vec{k}$, so $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = -3v^2\vec{i} - 2u\vec{j} + 6uv^2\vec{k}$

Since the point $(5, 2, 3)$ corresponds to $u = 2$, $v = 1$, a normal vector to the surface at $(5, 2, 3)$ is $-3\vec{i} - 4\vec{j} + 12\vec{k}$, and an equation of the tangent plane is $-3(x-5) - 4(y-2) + 12(z-3) = 0$ or $\boxed{3x + 4y - 12z = -13}$

(b) $\vec{r}(u, v) = \sin u\,\vec{i} + \cos u \sin v\,\vec{j} + \sin v\,\vec{k}$; $u = \frac{\pi}{6}$, $v = \frac{\pi}{6}$

Note that $\vec{r}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$

$\vec{r}_u = \cos u\,\vec{i} - \sin u \sin v\,\vec{j}$ and $\vec{r}_v = \cos u \cos v\,\vec{j} + \cos v\,\vec{k}$.

Hence, $\vec{r}_u\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\vec{i} - \frac{1}{4}\vec{j}$ and $\vec{r}_v\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{4}\vec{j} + \frac{\sqrt{3}}{2}\vec{k}$. A normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is

$\vec{r}_u\left(\frac{\pi}{6}, \frac{\pi}{6}\right) \times \vec{r}_v\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\sqrt{3}}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{\sqrt{3}}{2} \end{vmatrix} = -\frac{\sqrt{3}}{8}\vec{i} - \frac{3}{4}\vec{j} + \frac{3\sqrt{3}}{8}\vec{k}$

Thus an equation of the tangent plane at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is

$-\frac{\sqrt{3}}{8}\left(x - \frac{1}{2}\right) - \frac{3}{4}\left(y - \frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z - \frac{1}{2}\right) = 0$ or $\boxed{\sqrt{3}x + 6y - 3\sqrt{3}z = \frac{\sqrt{3}}{2}}$

4. Find the area of the surface.

(a) The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$.

Here we can write $z = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$ and D is the disk $x^2 + y^2 \le 3$. If we let x and y be parameters, then $x = x$, $y = y$, and $z = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$. Thus, $\vec{r}(x, y) = x\vec{i} + y\vec{j} + \left(\frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y\right)\vec{k}$. Then $\vec{r}_x = \vec{i} - \frac{1}{3}\vec{k}$ and $\vec{r}_y = \vec{j} - \frac{2}{3}\vec{k}$.

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{vmatrix} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \vec{k}$ and $|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{1}{9} + \frac{4}{9} + 1} = \frac{\sqrt{14}}{3}$. Then,

$A = \int\int_D \frac{\sqrt{14}}{3}\,dA = \frac{\sqrt{14}}{3} \cdot (\text{ area of } D) = \frac{\sqrt{14}}{3} \cdot \pi(\sqrt{3})^2 = \boxed{\sqrt{14}\pi}$

(b) The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2$.

$\vec{r}_u = 2u\vec{i} + v\vec{j}$ and $\vec{r}_v = u\vec{j} + v\vec{k}$, Then,

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = v^2\vec{i} - 2uv\vec{j} + 2u^2\vec{k}$ and $|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4}$

$A = \int\int_D \sqrt{v^4 + 4u^2v^2 + 4u^4}\,dA = \int_0^1 \int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\,dv\,du = \int_0^1 \int_0^2 \sqrt{(v^2 + 2u^2)^2}\,dv\,du$

$= \int_0^1 \int_0^2 (v^2 + 2u^2)\,dv\,du = \int_0^1 \left[\frac{1}{3}v^3 + 2u^2v\right]_0^2 du = \int_0^1 \left(\frac{8}{3} + 4u^2\right) du$

$= \left[\frac{8}{3}u + \frac{4}{3}u^3\right]_0^1 = \boxed{4}$

E.6 Surface Integrals

In this section we learn how to integrate functions over a three-dimensional surface S.

Definition. Let S be a surface in three-dimensional space with parametric representation $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ where (u,v) are points of a two-dimensional space D. Then, the **surface integral of f over the surface S** is given by

$$\boxed{\int\int_S f(x,y,z)\,dS = \int\int_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,dA}$$

Note. Surface integrals are the three-dimensional analog of line integrals. $\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\cdot|r'(t)|\,dt$

Fact. If $f(x,y,z) = 1$, then $\int\int_S 1\,dS =$ Surface area of S

Example. Evaluate the surface integral $\int\int_S (x+y+z)\,dS$, where S is the parallelogram with parametric equations $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$.

$\vec{r}(u,v) = (u+v)\vec{i} + (u-v)\vec{j} + (1+2u+v)\vec{k}$

$\vec{r}_u = \vec{i} + \vec{j} + 2\vec{k}$

$\vec{r}_v = \vec{i} - \vec{j} + \vec{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\vec{k}$$

$$= 3\vec{i} + \vec{j} - 2\vec{k}$$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$

$D = \{(u,v) \mid 0 \le u \le 2,\ 0 \le v \le 1\}$

$f(\vec{r}(u,v)) = u+v+u-v+1+2u+v = 4u+v+1$

$$\int\int_S x+y+z\,dS = \int\int_D (4u+v+1)\cdot\sqrt{14}\,dA = \sqrt{14}\int_0^1\int_0^2 4u+v+1\,du\,dv$$

$$= \sqrt{14}\int_0^1 \left[2u^2+uv+u\right]_0^2 dv = \sqrt{14}\int_0^1 8+2v+2-(0+0+0)\,dv$$

$$= \sqrt{14}\int_0^1 10+2v\,dv = \sqrt{14}\left[10v+v^2\right]_0^1 = \sqrt{14}\left[10+1\right] = \boxed{11\sqrt{14}}$$

Example. Evaluate the surface integral $\int\int_S xyz\,dS$, where S is the cone with parametric equations $x = u\cos v$, $y = u\sin v$, $z = u$, $0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$.

$\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + u\,\vec{k}$

$\vec{r}_u = \cos v\,\vec{i} + \sin v\,\vec{j} + \vec{k}$

$\vec{r}_v = -u\sin v\,\vec{i} + u\cos v\,\vec{j}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \begin{vmatrix} \sin v & 1 \\ u\cos v & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} \cos v & 1 \\ -u\sin v & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix}\vec{k}$$

$= -u\cos v\,\vec{i} - u\sin v\,\vec{j} + (u\cos^2 v + u\sin^2 v)\,\vec{k} = \langle -u\cos v, -u\sin v, u \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$ (u^2)

$D = \{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le \pi/2\}$

$f(\vec{r}(u,v)) = (u\cos v)(u\sin v)\cdot u = u^3\cos v\sin v$

$$\int\int_S xyz\,dS = \int\int_D u^3\cos v\sin v \cdot \sqrt{2}\,u\,dA = \sqrt{2}\int_0^1\int_0^{\pi/2} u^4\cos v\sin v\,dv\,du$$

$u = \sin v$, $du = \cos v\,dv$; $v = 0$, $u = \sin 0 = 0$; $v = \pi/2$, $u = \sin\pi/2 = 1$

$$= \sqrt{2}\cdot\int_0^1 u^4\,du \cdot \int_0^{\pi/2} \cos v\sin v\,dv$$

$$= \sqrt{2}\cdot\left[\frac{u^5}{5}\right]_0^1 \cdot \int_0^1 u\,du$$

$$= \sqrt{2}\cdot\frac{1}{5}\cdot\left[\frac{u^2}{2}\right]_0^1 = \boxed{\frac{\sqrt{2}}{10}}$$

Example. Evaluate the surface integral $\int\int_S x^2yz\,dS$, where S is the part of the plane $z = 1+2x+3y$ that lies above the rectangle $[0,3] \times [0,2]$.

$z = f(x,y)$

$x = x$

$y = y$

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (1+2x+3y)\vec{k}$

$\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle = \langle -2, -3, 1 \rangle$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{(-2)^2+(-3)^2+1^2} = \sqrt{14}$

$f(\vec{r}(x,y)) = x^2y(1+2x+3y) = x^2y + 2x^3y + 3x^2y^2$

$D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$

$$\iint_S x^2yz\,dS = \iint_D (x^2y + 2x^3y + 3x^2y^2)\sqrt{14}\,dA$$

$$= \sqrt{14}\int_0^3\int_0^2 x^2y + 2x^3y + 3x^2y^2\,dy\,dx$$

$$= \sqrt{14}\int_0^3 \left[\frac{x^2y^2}{2} + x^3y^2 + x^2y^3\right]_0^2 dx = \sqrt{14}\int_0^3 2x^2 + 4x^3 + 8x^2\,dx$$

$$= \sqrt{14}\int_0^3 10x^2 + 4x^3\,dx = \sqrt{14}\left[\frac{10x^3}{3} + x^4\right]_0^3$$

$$= \sqrt{14}\left[90 + 81\right] = \boxed{171\sqrt{14}}$$

$x=\rho\sin\phi\cos\theta$
$y=\rho\sin\phi\sin\theta$
$z=\rho\cos\phi$

Example. Evaluate the surface integral $\int\int_S y^2\,dS$, where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$.

$\rho^2=1 \Rightarrow \rho=1$

$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}=\sqrt{\rho^2\sin^2\phi}=\rho\sin\phi \Rightarrow \cos\phi=\sin\phi \Rightarrow \phi=\pi/4$

$\vec{r}(\phi,\theta)=\sin\phi\cos\theta\,\vec{i}+\sin\phi\sin\theta\,\vec{j}+\cos\phi\,\vec{k}$

$\vec{r}_\phi=\cos\phi\cos\theta\,\vec{i}+\cos\phi\sin\theta\,\vec{j}-\sin\phi\,\vec{k}$

$\vec{r}_\theta=-\sin\phi\sin\theta\,\vec{i}+\sin\phi\cos\theta\,\vec{j}$

$$\vec{r}_\phi\times\vec{r}_\theta=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \cos\phi\cos\theta&\cos\phi\sin\theta&-\sin\phi\\ -\sin\phi\sin\theta&\sin\phi\cos\theta&0\end{vmatrix}=\begin{vmatrix}\cos\phi\sin\theta&-\sin\phi\\ \sin\phi\cos\theta&0\end{vmatrix}\vec{i}-\begin{vmatrix}\cos\phi\cos\theta&-\sin\phi\\ -\sin\phi\sin\theta&0\end{vmatrix}\vec{j}$$

$$+\begin{vmatrix}\cos\phi\cos\theta&\cos\phi\sin\theta\\ -\sin\phi\sin\theta&\sin\phi\cos\theta\end{vmatrix}\vec{k}=\sin^2\phi\cos\theta\,\vec{i}+\sin^2\phi\sin\theta\,\vec{j}+(\sin\phi\cos\phi\cos^2\theta+\sin\phi\cos\phi\sin^2\theta)\vec{k}$$

$=\langle\sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi\rangle$

$$\left|\vec{r}_\phi\times\vec{r}_\theta\right|=\sqrt{\sin^4\phi\cos^2\theta+\sin^4\phi\sin^2\theta+\sin^2\phi\cos^2\phi}=\sqrt{\sin^4\phi+\sin^2\phi\cos^2\phi}$$

$$=\sqrt{\sin^2\phi\,(\sin^2\phi+\cos^2\phi)}=\sqrt{\sin^2\phi}=\sin\phi$$

$D=\{(\phi,\theta)\mid 0\le\phi\le\pi/4,\ 0\le\theta\le2\pi\}$

$$\iint_S y^2\,dS=\iint_D \sin^2\phi\sin^2\theta\cdot\sin\phi\,dA=\int_0^{2\pi}\int_0^{\pi/4}\sin^3\phi\sin^2\theta\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\sin^2\theta\,d\theta\cdot\int_0^{\pi/4}\sin^3\phi\,d\phi=\int_0^{2\pi}\frac{1}{2}(1-\cos2\theta)\,d\theta\cdot\int_0^{\pi/4}\sin^2\phi\sin\phi\,d\theta$$

$$=\int_0^{2\pi}\frac{1}{2}(1-\cos2\theta)\,d\theta\cdot\int_0^{\pi/4}(1-\cos^2\phi)\cdot\sin\phi\,d\phi$$

$u=\cos\phi$
$du=-\sin\phi\,d\phi$
$\phi=0,\ u=\cos0=1$
$\phi=\pi/4,\ u=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$$=\frac{1}{2}\left[\theta-\frac{\sin2\theta}{2}\right]_0^{2\pi}\cdot\int_1^{\sqrt{2}/2}-(1-u^2)\,du$$

$$=\frac{1}{2}\left[2\pi-0-0+0\right]\cdot\left[\frac{u^3}{3}-u\right]_1^{\sqrt{2}/2}=\pi\left[\frac{(\frac{\sqrt{2}}{2})^3}{3}-\frac{\sqrt{2}}{2}-\frac{1}{3}+1\right]$$

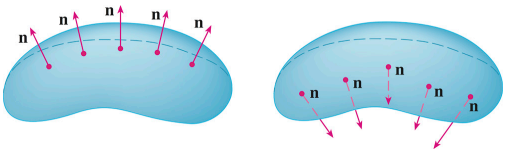
$$=\boxed{\pi\left(\frac{2}{3}-\frac{5\sqrt{2}}{12}\right)}$$

Surface Integrals of Vector Fields

Before we discuss surface integrals of vector fields, we need to introduce the concept of an oriented surface.

Suppose S is an orientable two-sided surface, which has a tangent plane at every point (x, y, z), except possible along the boundary. Then there are two unit normal vectors $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$ at any point (x, y, z). If it is possible to choose a unit normal vector $\vec{n}$ at every such point (x, y, z) so that $\vec{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\vec{n}$ provides S with an **orientation**.

Note. There are two possible orientations for any orientable surface.



Definition. A surface S is a **closed surface** if it is the boundary of some solid region E.

For example, a sphere is a closed surface. (It's the boundary of a ball)

Definition. We say that the closed surface S has a **positive** orientation if we choose the set of unit normal vectors that point outward from the region E while the **negative** orientation will be the set of unit normal vectors that point in towards the region E.

Definition. Given a vector field $\vec{F}$ with unit normal vector $\vec{n}$ then the surface integral of $\vec{F}$ over the oriented surface S is given by

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \vec{F} \cdot \vec{n}\, dS$$

Assuming positive orientation

and it is called the **flux of** $\vec{F}$ **across** S. It measures how much of $\vec{F}$ is passing through S.

- If S is given by a vector function $\vec{r}(u, v)$, then

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

(dot product)

- If $z = g(x, y)$ and $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, then

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

Assuming positive orientation

Example. Find the flux of $\vec{F}$ across S for $\vec{F}(x,y,z) = ze^{xy}\vec{i} - 3ze^{xy}\vec{j} + xy\vec{k}$, where S is the parallelogram with parametric equations $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$ with upward orientation.

positive

$\vec{r}(u,v) = (u+v)\vec{i} + (u-v)\vec{j} + (1+2u+v)\vec{k}$

We found in a previous example that $\vec{r}_u \times \vec{r}_v = 3\vec{i} + \vec{j} - 2\vec{k}$

$\vec{F}(\vec{r}(u,v)) = (1+2u+v)e^{(u+v)(u-v)}\vec{i} - 3(1+2u+v)e^{(u+v)(u-v)}\vec{j} + (u+v)(u-v)\vec{k}$

$= (1+2u+v)e^{u^2-v^2}\vec{i} - 3(1+2u+v)e^{u^2-v^2}\vec{j} + (u^2-v^2)\vec{k}$

$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 3(1+2u+v)e^{u^2-v^2} - 3(1+2u+v)e^{u^2-v^2} - 2(u^2-v^2)$

$= -2(u^2-v^2)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA = -\iint_D -2(u^2-v^2)\,dA$$

$D = \{(u,v) \mid 0 \le u \le 2,\ 0 \le v \le 1\}$

$$= -\int_0^1\int_0^2 -2(u^2-v^2)\,du\,dv = 2\int_0^1 \left[\frac{u^3}{3} - uv^2\right]_0^2 dv$$

$$= 2\int_0^1 \frac{8}{3} - 2v^2\,dv = 2\left[\frac{8}{3}v - \frac{2}{3}v^3\right]_0^1 = 2\left[\frac{8}{3} - \frac{2}{3}\right]$$

$$= 2\cdot\frac{6}{3} = \boxed{4}$$

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

Example. Find the flux of the vector field $\vec{F}(x,y,z) = z\vec{i} + y\vec{j} + x\vec{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

$\rho^2 = 1 \Rightarrow \rho = 1$

$\vec{r}(\phi,\theta) = \sin\phi\cos\theta\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \cos\phi\,\vec{k}$

We found in a previous example that

$\vec{r}_\phi \times \vec{r}_\theta = \langle \sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi \rangle$

$\vec{F}(\vec{r}(\phi,\theta)) = \cos\phi\,\vec{i} + \sin\phi\sin\theta\,\vec{j} + \sin\phi\cos\theta\,\vec{k}$

$\vec{F}(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = \sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta + \sin^2\phi\cos\phi\cos\theta$
$= 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta$

$D = \{(\phi,\theta) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$

$$\iint_S \vec{F}\,d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta\,d\phi\,d\theta + \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\sin^2\theta\,d\phi\,d\theta$$

$$= 2\int_0^{2\pi}\cos\theta\,d\theta \cdot \int_0^{\pi}\sin^2\phi\cos\phi\,d\phi + \int_0^{2\pi}\sin^2\theta\,d\theta \cdot \int_0^{\pi}\sin^3\phi\,d\phi$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin^3\phi = (1-\cos^2\theta)\sin\theta$

$u = \sin\phi$, $du = \cos\phi\,d\phi$

$u = \cos\theta$, $du = -\sin\theta\,d\theta$

$$= 2\cdot\left[\sin\theta\right]_0^{2\pi} \cdot \int_{\phi=0}^{\phi=\pi} u^2\,du + \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{2\pi} \cdot \int_1^{-1} -(1-u^2)\,du$$

0

$$= \frac{1}{2}\left[2\pi - 0 - 0 + 0\right] \cdot \left[u - \frac{u^3}{3}\right]_{-1}^{1}$$

$$= \pi\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] = \boxed{\frac{4\pi}{3}}$$

Example. Find the flux of the vector field $\vec{F}(x,y,z) = -x\vec{i} - y\vec{j} + z^3\vec{k}$, S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation.

$z = g(x,y)$ $\iint_S \vec{F}\, d\vec{S} = -\iint_D (-Pg_x - Qg_y + R)\, dA$

$P = -x$, $Q = -y$, $R = z^3$

$g_x = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$

$g_y = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$

annular ring

$D = \{(x,y) \mid 1 \le \sqrt{x^2+y^2} \le 3\} = \{(x,y) \mid 1 \le x^2+y^2 \le 9\} = \{(r,\theta) \mid 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_D \left[-(-x)\cdot\frac{x}{\sqrt{x^2+y^2}} - (-y)\cdot\frac{y}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right] dA$$

$$= -\iint_D \left[\frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right] dA$$

switch to polar: $r^2 = x^2+y^2$

$$= -\int_0^{2\pi}\int_1^3 \left(\frac{r^2}{\sqrt{r^2}} + r^3\right) r\, dr\, d\theta = -\int_0^{2\pi} d\theta \cdot \int_1^3 r^2 + r^4\, dr$$

$$= -[2\pi - 0]\cdot\left[\frac{r^3}{3} + \frac{r^5}{5}\right]_1^3 = -2\pi\left(9 + \frac{243}{5} - \frac{1}{3} - \frac{1}{5}\right)$$

$$= \boxed{-\frac{1712}{15}\pi}$$

Homework E.6. - Surface Integrals

Prof. Leyda Almodóvar

Name: _____

1. Evaluate the surface integral.

   (a) $\int\int_S y\,dS$, S is the helicoid with vector equation $\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$.

   $\vec{r}_u = \langle \cos v, \sin v, 0\rangle$ and $\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$. Then,

   $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\rangle$$

   $$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

   $$\int\int_S y\,dS = \int\int_D (u\sin v)|\vec{r}_u \times \vec{r}_v|\,dA = \int_0^1\int_0^\pi (u\sin v)\sqrt{u^2+1}\,dv\,du$$

   $$= \int_0^1 u\sqrt{u^2+1}\,du \int_0^\pi \sin v\,dv = \left[\tfrac{1}{3}(u^2+1)^{\frac{3}{2}}\right]_0^1 \cdot \Big[-\cos v\Big]_0^\pi$$

   $$= \tfrac{1}{3}(2^{\frac{3}{2}} - 1)\cdot 2 = \boxed{\frac{2}{3}(2\sqrt{2}-1)}$$

   (b) $\int\int_S (x^2+y^2)\,dS$, S is the surface with vector equation $\vec{r}(u,v) = \langle 2uv, u^2 - v^2, u^2+v^2\rangle$, $u^2+v^2 \le 1$.

   $\vec{r}_u = \langle 2v, 2u, 2u\rangle$ and $\vec{r}_v = \langle 2u, -2v, 2v\rangle$. Then,

   $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2v & 2u & 2u \\ 2u & -2v & 2v \end{vmatrix} = \langle 8uv, 4u^2 - 4v^2, -4u^2 - 4v^2\rangle$$

   $$|\vec{r}_u \times \vec{r}_v| = \sqrt{(8uv)^2 + (4u^2-4v^2)^2 + (-4u^2-4v^2)^2} = \sqrt{64u^2v^2 + 32u^4 + 32v^4}$$

   $$= \sqrt{32(u^2+v^2)^2} = 4\sqrt{2}(u^2+v^2)$$

$\int\int_S (x^2+y^2)\,dS = \int\int_D [(2uv)^2+(u^2-v^2)^2]|\vec{r}_u \times \vec{r}_v|\,dA$

$= \int\int_D (4u^2v^2+u^4-2u^2v^2+v^4)\cdot 4\sqrt{2}(u^2+v^2)\,dA$

$= 4\sqrt{2}\int\int_D (u^4+2u^2v^2+v^4)(u^2+v^2)\,dA = 4\sqrt{2}\int\int_D (u^2+v^2)^3\,dA = 4\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2)^3 r\,dr\,d\theta$

$= 4\sqrt{2}\int_0^{2\pi} d\theta \int_0^1 r^7\,dr = 4\sqrt{2}\Big[\theta\Big]_0^{2\pi}\Big[\tfrac{1}{8}r^8\Big]_0^1 = 4\sqrt{2}\cdot 2\pi\cdot\tfrac{1}{8} = \boxed{\sqrt{2}\pi}$

(c) $\int\int_S (x^2z+y^2z)\,dS$, S is the hemisphere $x^2+y^2+z^2=4$, $z\geq 0$.

Using spherical coordinates to parametrize the sphere we have
$\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$. Then,
$r_\phi = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi\rangle$ and $r_\theta = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0\rangle$

$$\vec{r}_\phi\times\vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} = \langle 4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$$

$|\vec{r}_\phi\times\vec{r}_\theta| = \sqrt{(4\sin^2\phi\cos\theta)^2+(-4\sin^2\phi\sin\theta)^2+(4\sin\phi\cos\phi)^2}$

$= \sqrt{16\sin^2\phi[\sin^2\phi(\cos^2\theta+\sin^2\theta)+\cos^2\phi]} = \sqrt{16\sin^2\phi(\sin^2\phi+\cos^2\phi)} = 4\sin\phi$

S is the portion of the sphere corresponding to $0\leq\phi\leq\frac{\pi}{2}$, so

$\int\int_S (x^2z+y^2z)\,dS = \int\int_S (x^2+y^2)z\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta$

$= 32\int_0^{2\pi} d\theta\int_0^{\frac{\pi}{2}} \sin^3\phi\cos\phi\,d\phi = 32(2\pi)\Big[\tfrac{1}{4}\sin^4\phi\Big]_0^{\frac{\pi}{2}} = 16\pi(1-0) = \boxed{16\pi}$

2. Find the flux of $\vec{F}$ across S. For closed surfaces, use the positive (outward) orientation.

(a) $\vec{F}(x,y,z) = z\vec{i}+y\vec{j}+x\vec{k}$, S is the helicoid from Problem 1 with upward orientation.

We found $\vec{r}_u\times\vec{r}_v = \langle\sin v, -\cos v, u\rangle$. Then $\vec{F}(\vec{r}(u,v)) = v\vec{i}+u\sin v\vec{j}+u\cos v\vec{k}$ and

$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA = \int_0^1\int_0^\pi (v\sin v - u\sin v\cos v + u^2\cos v)\,dv\,du$

$= \int_0^1 [\sin v - v\cos v - \tfrac{1}{2}u\sin^2 v + u^2\sin v\Big]_0^\pi du = \int_0^1 \pi\,du = [\pi u]_0^1 = \boxed{\pi}$

(b) $\vec{F}(x,y,z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, and has upward orientation.

Note that $z = g(x,y)$, $P = xy$, $Q = yz$, $R = zx$, $g_x = -2x$, $g_y = -2y$. Then,

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_D \Big(-Pg_x - Qg_y + R\Big)\,dA = \int\int_D [-xy(-2x) - yz(-2y) + zx]\,dA$$

$$= \int_0^1 \int_0^1 [2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2)]\,dy\,dx$$

$$= \int_0^1 \left[x^2y^2 + \tfrac{8}{3}y^3 - \tfrac{2}{3}x^2y^3 - \tfrac{2}{5}y^5 + 4xy - x^3y - \tfrac{1}{3}xy^3\right]_0^1 dx$$

$$= \int_0^1 \left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right) dx = \left[\tfrac{1}{9}x^3 + \tfrac{11}{6}x^2 - \tfrac{1}{4}x^4 + \tfrac{34}{15}x\right]_0^1 = \boxed{\frac{713}{180}}$$

(c) $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z^2\vec{k}$, S is the sphere with radius 1 and center the origin.

Using spherical coordinates, the vector equation for S is $\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.
$\vec{F}(\vec{r}(\phi,\theta)) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos^2\phi\rangle$

$r_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi\rangle$ and $r_\theta = \langle \sin\phi\sin\theta, -\sin\phi\cos\theta, 0\rangle$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ \sin\phi\sin\theta & -\sin\phi\cos\theta & 0 \end{vmatrix} = \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\rangle$$

$$\vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi \times \vec{r}_\theta) = \sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^3\phi$$

$$= \sin^3\phi + \sin\phi\cos^3\phi$$

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_D [\vec{F}(\vec{r}(\phi,\theta))\cdot(\vec{r}_\phi \times \vec{r}_\theta)]\,dA = \int_0^{2\pi}\int_0^{\pi} (\sin^3\phi + \sin\phi\cos^3\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} (1 - \cos^2\phi + \cos^3\phi)\sin\phi\,d\phi = (2\pi)\Big[-\cos\phi + \tfrac{1}{3}\cos^3\phi - \tfrac{1}{4}\cos^4\phi\Big]_0^{\pi}$$

$$= 2\pi\left(1 - \tfrac{1}{3} - \tfrac{1}{4} + 1 - \tfrac{1}{3} + \tfrac{1}{4}\right) = \boxed{\frac{8}{3}\pi}$$

E.7 Stokes' Theorem

In this section we study Stokes' Theorem, a higher-dimensional version of Green's Theorem. While Green's Theorem relates a double integral to a line integral, Stokes' Theorem relates a surface integral to a line integral.

Theorem (Stokes' Theorem). Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives. Then,

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}}$$

↓ counterclockwise when viewed from above

Stokes' Theorem states that the line integral around the boundary curve of S of the tangential components of $\vec{F}$ is equal to the surface integral over S of the normal component of the curl of $\vec{F}$.

Example. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$ for $\vec{F}(x,y,z) = x^2 \sin z\,\vec{i} + y^2\vec{j} + xy\vec{k}$, S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward.

$z \geq 0$

The paraboloid $z = 1 - x^2 - y^2$ intersects the xy-plane in the circle $x^2 + y^2 = 1$, $z = 0$.
This boundary C must be oriented counterclockwise when viewed from above.
The vector equation of C is $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$, $0 \leq t \leq 2\pi$.

$\vec{F}(\vec{r}(t)) = \cos^2 t \cdot \sin 0\,\vec{i} + \sin^2 t\,\vec{j} + \cos t \sin t\,\vec{k} = \sin^2 t\,\vec{j} + \cos t \sin t\,\vec{k}$

$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0 + \cos t \sin^2 t + 0 = \cos t \sin^2 t$

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \cos t \sin^2 t\,dt$$

$u = \sin t$
$du = \cos t\,dt$

$$= \int_0^0 u^2\,du = \boxed{0}$$

Example. Use Stokes' Theorem to evaluate $\int\int_S \text{curl } \vec{F} \cdot d\vec{S}$ for $\vec{F}(x,y,z) = \tan^{-1}(x^2yz^2)\vec{i} + x^2y\vec{j} + x^2z^2\vec{k}$, S is the cone $x = \sqrt{y^2+z^2}$, $0 \le x \le 2$, oriented in the direction of the positive x-axis.

The boundary curve C is $2 = \sqrt{y^2+z^2} \Rightarrow 4 = y^2+z^2$, $x=2$, a circle, which should be oriented counterclockwise when viewed from above. A vector equation of C is $\vec{r}(t) = 2\vec{i} + 2\cos t\,\vec{j} + 2\sin t\,\vec{k}$, $0 \le t \le 2\pi$.

$\vec{F}(\vec{r}(t)) = \tan^{-1}(4 \cdot 2\cos t \cdot 4\sin^2 t)\,\vec{i} + 4 \cdot 2\cos t\,\vec{j} + 4 \cdot 4\sin^2 t\,\vec{k}$

$= \tan^{-1}(32\cos t \sin^2 t)\,\vec{i} + 8\cos t\,\vec{j} + 16\sin^2 t\,\vec{k}$

$\vec{r}\,'(t) = 0\vec{i} - 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -16\sin t\cos t + 32\sin^2 t\cos t$

$\int\int_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi} (-16\sin t\cos t + 32\sin^2 t\cos t)\,dt$

$u = \sin t$, $du = \cos t\,dt$

$= \int_0^0 -16u + 32u^2\,du = \boxed{0}$

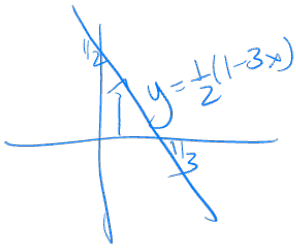
Example. Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z)=\vec{i}+(x+yz)\vec{j}+(xy-\sqrt{z})\vec{k}$, where C is the boundary of the part of the plane $3x+2y+z=1$ in the first octant oriented counterclockwise as viewed from above.

$z = 1-3x-2y = g(x,y)$ $\quad g_x=-3$ $\quad g_y=-2$

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\,dA$ where $\operatorname{curl}\vec{F} = \langle P, Q, R\rangle$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = \left[\frac{\partial}{\partial y}(xy-\sqrt{z}) - \frac{\partial}{\partial z}(x+yz)\right]\vec{i} - \left[\frac{\partial}{\partial x}(xy-\sqrt{z}) - \frac{\partial}{\partial z}(1)\right]\vec{j} + \left[\frac{\partial}{\partial x}(x+yz) - \frac{\partial}{\partial y}(1)\right]\vec{k} = (x-y)\vec{i} - y\vec{j} + \vec{k}$$

$$= \int\int_D [(x-y)(-3) - (-y)(-2) + 1]\,dA = \int_0^{1/3}\int_0^{\frac{1}{2}(1-3x)} 3x - 3y - 2y + 1\,dy\,dx$$



$$= \int_0^{1/3}\int_0^{\frac{1}{2}(1-3x)} 3x - 5y + 1\,dy\,dx$$

$$= \int_0^{1/3} 3xy - \frac{5y^2}{2} + y\Big]_0^{\frac{1}{2}(1-3x)}\,dx$$

$D = \{(x,y) \mid 0 \le y \le \frac{1}{2}(1-3x),\ 0 \le x \le 1/3\}$

$$= \int_0^{1/3} 3x\cdot\frac{1}{2}(1-3x) - \frac{5}{2}\cdot\frac{1}{4}(1-3x)^2 + \frac{1}{2}(1-3x)\,dx$$

$$= \int_0^{1/3} -\frac{81}{8}x^2 + \frac{15}{4}x - \frac{1}{8}\,dx = -\frac{27}{8}x^3 + \frac{15}{8}x^2 - \frac{1}{8}x\Big]_0^{1/3}$$

$$= -\frac{27}{8}\cdot\frac{1}{27} + \frac{15}{8}\cdot\frac{1}{9} - \frac{1}{8}\cdot\frac{1}{3} = -\frac{1}{8} + \frac{5}{24} - \frac{1}{24} = \boxed{\frac{1}{24}}$$

Homework E.7 - Stokes' theorem

Prof. Leyda Almodóvar

Name: _____

1. Use Stokes' Theorem to evaluate $\int\int_S \text{ curl } \vec{F} \cdot d\vec{S}$.

   (a) $\vec{F}(x,y,z) = ze^y\vec{i} + x\cos y\vec{j} + xz\sin y\vec{k}$, S is the hemisphere $x^2+y^2+z^2=16$, $y \le 0$, oriented in the direction of the positive y-axis.

   The boundary C is the circle $x^2+z^2=16$, $y=0$ where the hemisphere intersects the xz-plane. The curve should be oriented in the counterclockwise direction when viewed from the right (from the positive y-axis), so a vector equation of C is $r(t) = 4\cos(-t)\vec{i} + 4\sin(-t)\vec{k} = 4\cos t\vec{i} - 4\sin t\vec{k}$, $0 \le t \le 2\pi$. Then, $r'(t) = -4\sin t\vec{i} - 4\cos t\vec{k}$ and $\vec{F}(\vec{r}(t)) = (-4\sin t)e^0\vec{i} + (4\cos t)(\cos 0)\vec{j} + (4\cos t)(-4\sin t)(\sin 0)\vec{k} = -4\sin t\vec{i} + 4\cos t\vec{j}$ and by Stokes' Theorem

   $\int\int_S \text{ curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$

   $= \int_0^{2\pi} (-4\sin t\vec{i} + 4\cos t\vec{j}) \cdot (-4\sin t\vec{i} - 4\cos t\vec{k})\,dt = \int_0^{2\pi} (16\sin^2 t + 0 + 0)\,dt$

   $= 16\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} = \boxed{16\pi}$

   (b) $\vec{F}(x,y,z) = e^{xy}\vec{i} + e^{xz}\vec{j} + x^2z\sin y\vec{k}$, S is the half of the ellipsoid $4x^2+y^2+4z^2=4$, that lies to the right of the xz-plane, oriented in the direction of the positive y-axis.

   The boundary C is the circle $x^2+z^2=1$, $y=0$ which should be oriented in the counterclockwise direction when viewed from the right, so a vector equation is $\vec{r}(t) = \cos(-t)\vec{i} + \sin(-t)\vec{k} = \cos t\vec{i} - \sin t\vec{k}$, $0 \le t \le 2\pi$. Then, $\vec{F}(\vec{r}(t)) = \vec{i} + e^{-\cos t \sin t}\vec{j} - \cos^2 t \sin t\vec{k}$, $\vec{r}'(t) = -\sin t\vec{i} - \cos t\vec{k}$, and $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -\sin t + \cos^3 t \sin t$. Thus

   $\int\int_S \text{ curl } \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_0^{2\pi} (-\sin t + \cos^3 t \sin t)\,dt$

   $= \left[\cos t - \frac{1}{4}\cos^4 t\right]_0^{2\pi} = \boxed{0}$

2. Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$. In each case C is oriented counterclockwise as viewed from above.

(a) $\vec{F}(x,y,z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

First we find curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = (0-y)\vec{i} - (z-0)\vec{j} + (0-x)\vec{k} = -y\vec{i} - z\vec{j} - x\vec{k}$

We take S to be the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant. Since C is oriented counterclockwise (from above), we orient S upward. Then $z = g(x,y) = 1 - x^2 - y^2$ and

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{ curl } \vec{F}\cdot d\vec{S} = \iint_D [-(-y)(-2x) - (-z)(-2y) + (-x)]\, dA$

$= \iint_D [-2xy - 2y(1-x^2-y^2) - x]\, dA$

$= \int_0^{\frac{\pi}{2}} \int_0^1 [-2(r\cos\theta)(r\sin\theta) - 2(r\sin\theta)(1-r^2) - r\cos\theta] r\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^1 [-2r^3 \sin\theta\cos\theta - 2(r^2 - r^4)\sin - r^2\cos\theta]\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \left[-\tfrac{1}{2}r^4 \sin\theta\cos\theta - 2\left(\tfrac{1}{3}r^3 - \tfrac{1}{5}r^5\right)\sin\theta - \tfrac{1}{3}r^3\cos\theta \right]_0^1 d\theta$

$= \int_0^{\frac{\pi}{2}} \left(-\tfrac{1}{2}\sin\theta\cos\theta - \tfrac{4}{15}\sin\theta - \tfrac{1}{3}\cos\theta \right) d\theta = \left[-\tfrac{1}{4}\sin^2\theta + \tfrac{4}{15}\cos\theta - \tfrac{1}{3}\sin\theta \right]_0^{\frac{\pi}{2}}$

$= -\tfrac{1}{4} - \tfrac{4}{15} - \tfrac{1}{3} = \boxed{-\dfrac{17}{20}}$

(b) $\vec{F}(x,y,z) = 2y\vec{i} + xz\vec{j} + (x+y)\vec{k}$, C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$.

The curve of intersection is an ellipse in the plane $z = y + 2$. Then,

curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & xz & x+y \end{vmatrix} = (1-x)\vec{i} - (1-0)\vec{j} + (z-2)\vec{k} = (1-x)\vec{i} - \vec{j} + (z-2)\vec{k}$

We take the surface S to be the planar region enclosed by C with upward orientation. Then, $z = g(x,y) = y + 2$ we have

$= \oint \vec{F}\cdot d\vec{r} = \iint_S \text{ curl } \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} [-(1-x)(0) - (-1)(1) + (y+2-2)]\, dA$

$= \iint_{x^2+y^2\le 1} (y+1)\, dA = \int_0^{2\pi}\int_0^1 (r\sin\theta + 1) r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}r^3\sin\theta + \tfrac{1}{2}r^2 \right]_0^1 d\theta$

$= \int_0^{2\pi} \left(\tfrac{1}{3}\sin\theta + \tfrac{1}{2} \right) d\theta = \left[-\tfrac{1}{3}\cos\theta + \tfrac{1}{2}\theta \right]_0^{2\pi} = \boxed{\pi}$

E.8 The Divergence Theorem

In this section we will relate surface integrals to triple integrals via the Divergence Theorem.

Theorem. Divergence Theorem Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives. Then,

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_E \text{div } \vec{F}\, dV$$

Triple Integral

Example. Use the Divergence Theorem to calculate the surface integral $\int\int_S \vec{F} \cdot d\vec{S}$, that is, calculate the flux of $\vec{F}$ across S, for $\vec{F}(x,y,z) = xye^z\vec{i} + xy^2z^3\vec{j} - ye^z\vec{k}$, S is the surface of the box bounded by the coordinates planes and the planes $x = 3$, $y = 2$, and $z = 1$.

$$E = \{(x,y,z) \mid 0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1\}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z)$$

$$= ye^z + 2xyz^3 - ye^z = 2xyz^3$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^3\int_0^2\int_0^1 2xyz^3\, dz\, dy\, dx = 2\int_0^3 x\, dx \cdot \int_0^2 y\, dy \cdot \int_0^1 z^3\, dz$$

$$= 2\left[\frac{x^2}{2}\right]_0^3 \cdot \left[\frac{y^2}{2}\right]_0^2 \cdot \left[\frac{z^4}{4}\right]_0^1$$

$$= 9 \cdot 2 \cdot \frac{1}{4} = \boxed{\frac{9}{2}}$$

Example. Use the Divergence Theorem to calculate the surface integral $\int\int_S \vec{F}\cdot d\vec{S}$, that is, calculate the flux of $\vec{F}$ across S, for $\vec{F}(x,y,z) = 3xy^2\vec{i} + xe^z\vec{j} + z^3\vec{k}$, S is the surface of the solid bounded by the cylinder $y^2+z^2=1$ and the planes $x=-1$ and $x=2$.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial y}(xe^z) + \frac{\partial}{\partial z}(z^3) = 3y^2 + 0 + 3z^2$$
$$= 3y^2 + 3z^2$$

$x = x$
$y = r\cos\theta$
$z = r\sin\theta$

$-1 \le x \le 2$

$r^2 = y^2 + z^2 \Rightarrow 0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 3y^2 + 3z^2\, dV$$
$$= \int_0^{2\pi}\int_0^1\int_{-1}^{2} (3r^2\cos^2\theta + 3r^2\sin^2\theta)\, r\, dx\, dr\, d\theta$$
$$= 3\int_0^{2\pi} d\theta \cdot \int_0^1 r^3\, dr \cdot \int_{-1}^{2} dx$$
$$= 3\cdot 2\pi\left[\frac{r^4}{4}\right]_0^1 \cdot [2+1] = 6\pi\cdot\frac{1}{4}\cdot 3 = \boxed{\frac{9\pi}{2}}$$

Homework E.8 -The Divergence Theorem

Prof. Leyda Almodóvar

Name: _____

1. Use the Divergence Theorem to calculate the surface integral $\int\int_S \vec{F}\cdot d\vec{S}$, that is, calculate the flux of $\vec{F}$ across S.

   (a) $\vec{F}(x,y,z) = (x^3+y^3)\vec{i}+(y^3+z^3)\vec{j}+(z^3+x^3)\vec{k}$, S is the sphere with center the origin and radius 2.

   div $\vec{F} = 3x^2+3y^2+3z^2$, so by the Divergence Theorem,

   $\int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_E 3(x^2+y^2+z^2)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^2 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

   $= 3\int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^4\,d\rho = 3\Big[-\cos\phi\Big]_0^{\pi}\Big[\theta\Big]_0^{2\pi}\Big[\frac{1}{5}\rho^5\Big]_0^2$

   $= 3(2)(2\pi)\left(\frac{32}{5}\right) = \boxed{\frac{384}{5}\pi}$

   (b) $\vec{F}(x,y,z) = xe^y\vec{i}+(z-e^y)\vec{j}-xy\vec{k}$, S is the ellipsoid $x^2+2y^2+3z^2=4$.

   div $\vec{F} = e^y-e^y+0 = 0$, so by the Divergence Theorem,

   $\int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_E 0\,dV = \boxed{0}$

   (c) $\vec{F}(x,y,z) = (2x^3+y^3)\vec{i}+(y^3+z^3)\vec{j}+3y^2z\vec{k}$, S is the surface of the solid bounded by the paraboloid $z = 1-x^2-y^2$ and the xy-plane.

   div $\vec{F} = 6x^2+3y^2+3y^2 = 6x^2+6y^2$ so

   $\int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_E 6(x^2+y^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6r^2\cdot r\,dz\,dr\,d\theta$

   $= \int_0^{2\pi}\int_0^1 6r^3(1-r^2)\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^1(6r^3-6r^5)\,dr$

   $= \Big[\theta\Big]_0^{2\pi}\Big[\frac{3}{2}r^4-r^6\Big]_0^1 = 2\pi\left(\frac{3}{2}-1\right) = \boxed{\pi}$

(d) $\vec{F}(x,y,z) = (xy+2xz)\vec{i} + (x^2+y^2)\vec{j} + (xy-z^2)\vec{k}$, S is the surface of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=y-2$ and $z=0$.

For $x^2+y^2 \le 4$ the plane $z=y-2$ is below the xy-plane, so the solid E bounded by S is $E = \{(x,y,z) | x^2+y^2 \le 4, y-2 \le z \le 0\}$.

div $\vec{F} = y+2z+2y-2z = 3y$ so

$\int\int_S \vec{F}\cdot d\vec{S} = \int\int\int_E 3y\, dV = \int_0^{2\pi}\int_0^2 \int_{r\sin\theta-2}^0 (3r\sin\theta) r\, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^2 (3r^2\sin\theta)(0 - r\sin\theta + 2)\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (-3r^3\sin^2\theta + 6r^2\sin\theta)\, dr\, d\theta$

$= \int_0^{2\pi} \left[-\frac{3}{4}r^4\sin^2\theta + 2r^3\sin\theta \right]_0^2 = \int_0^{2\pi} (-12\sin^2\theta + 16\sin\theta)\, d\theta$

$= \left[-12\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) - 16\cos\theta\right]_0^{2\pi} = -12\pi - 16 + 16 = \boxed{-12\pi}$

Practice Exam 1 MTH-261

Prof. Leyda Almodóvar

Name: _____

Answer the questions in the spaces provided on thequestion sheets. If you run out of room for an answer, you may use an additional blank sheet. You must show your work on every problem. Answers with no work or justification shown will get no credit.

Question	Points	Score
1	12	
2	10	
3	10	
4	8	
5	20	
6	6	
7	10	
8	12	
9	12	
Total:	100	

1. (12 points) Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

2. (10 points) Describe in words the region of $\mathbb{R}^3$ represented by the equation(s) or inequality.

(a) $x^2 + y^2 \leq 7$

(b) $0 \leq y \leq 5$

3. (10 points) Find an equation of the plane through the origin and the points $(3, -2, 1)$ and $(1, 1, 1)$.

4. (8 points) Given $P = (0, 0, -3)$, $Q = (4, 2, 0)$, and $R = (3, 3, 1)$,

(a) find a nonzero vector orthogonal to the plane through the points P, Q, and R, and

(b) find the area of triangle PQR.

5. (20 points) Given $\vec{a} = \langle 4, -1, 3 \rangle$ and $\vec{b} = \langle 5, 2, -7 \rangle$, find

(a) $|\vec{a}|$

(b) $|\vec{a} - \vec{b}|$

(c) $\vec{a} \cdot \vec{b}$

(d) $\vec{a} \times \vec{b}$

(e) Show $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

(f) the angle between $\vec{a}$ and $\vec{b}$

(g) a unit vector $\vec{u}$ with the same direction as $\vec{b}$

(h) the vector projection of $\vec{b}$ onto $\vec{a}$

6. (6 points) Determine whether the lines L_1: $x = 5 - 12t$, $y = 3 + 9t$, $z = 1 - 3t$, L_2: $x = 3 + 8s$, $y = -6s$, $z = 7 + 2s$ are parallel, skew, or intersecting. If they intersect, find the point of intersection.

7. (10 points) Find an equation of the plane through the points $(2, 1, 2)$, $(3, -8, 6)$, and $(-2, -3, 1)$. Simplify the equation to be in the form $ax + by + cz = d$ for some constants a, b, c, d.

8. (12 points) **Reduce** the equation $x^2 + y^2 - 2x - 6y - z + 10 = 0$ to one of the standard forms, **classify** the surface, and **sketch** it. **Clearly state the vertex and the axis of the quadric surface.**

9. (12 points) (a) Find and describe the horizontal and vertical traces of $f(x, y) = ye^x$.

(b) Sketch the vertical traces of f in 2-space.

(c) Draw a contour map of the function showing three fixed curves.

Review Exam 1

B 5 26) $x^2+y^2-2x-6y-z+10=0$

and in Practice Exam

$x^2-2x+\left(\frac{-2}{2}\right)^2-\left(\frac{-2}{2}\right)^2+y^2-6y+\left(\frac{-6}{2}\right)^2-\left(\frac{-6}{2}\right)^2-z+10=0$

$(x-1)^2$ $\qquad$ $(y-3)^2$

$(x-1)^2-1+(y-3)^2-9-z+10=0$

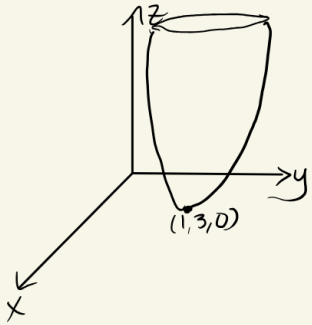
$(x-1)^2+(y-3)^2-z=0$

$(x-1)^2+(y-3)^2=z$

elliptic paraboloid

Vertex is $(1,3,0)$

The axis is the line $x=1$ and $y=3$



#4 on P.E. Given $P=(0,0,-3)$, $Q=(4,2,0)$, $R=(3,3,1)$

(a) Find a nonzero vector orthogonal to the plane through P, Q, and R.

$\vec{PQ} = \langle 4-0, 2-0, 0+3\rangle = \langle 4,2,3\rangle$

$\vec{PR} = \langle 3-0, 3-0, 1+3\rangle = \langle 3,3,4\rangle$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix}\vec{k}$$

$$= (2\cdot 4 - 3\cdot 3)\vec{i} - (16-9)\vec{j} + (12-6)\vec{k}$$

$$= -\vec{i} - 7\vec{j} + 6\vec{k}$$

(b) Find the Area of the triangle PQR.

Area of $\triangle PQR$ is $\frac{|\langle -1,-7,6\rangle|}{2}$

$$= \frac{\sqrt{(-1)^2+(-7)^2+6^2}}{2}$$

$$= \frac{\sqrt{1+49+36}}{2} = \boxed{\frac{\sqrt{86}}{2}}$$

#9 on P.E. (a) Find and describing the horizontal and vertical traces of $f(x,y) = ye^x$.

Vertical traces

- $x = K$: $z = ye^K$ $\Rightarrow$ lines with positive slope ($e^K > 0$)
  $K = 0$, $z = y$
- $y = K$: $z = Ke^x$
  If $K > 0$, we obtain increasing exponential functions
  If $K < 0$, decreasing exp. functions
  If $K = 0$, $z = 0$ (xy-plane)

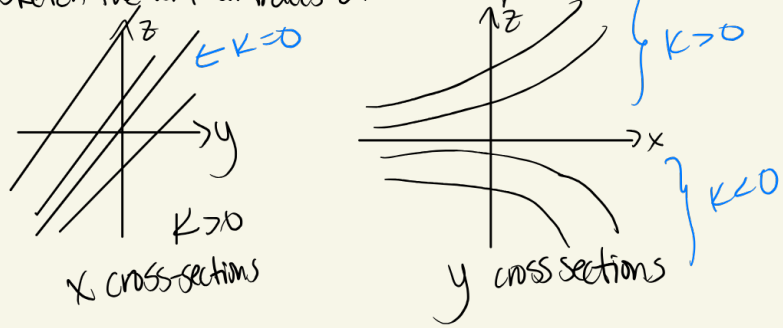
Horizontal traces

$z = K$: $K = ye^x$

$y = Ke^{-x}$

exponential functions
decreasing if $K > 0$
increasing if $K < 0$

(b) Sketch the vertical traces of f in z-space.



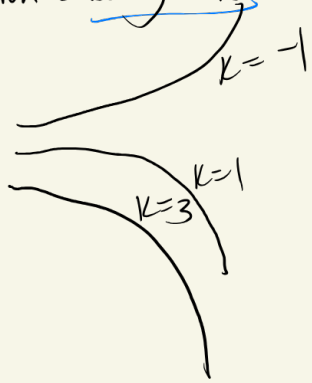
(c) Draw a contour map of the function showing three fixed curves.

Fix K for 3 different values

$K = 1$: $1 = ye^x \Rightarrow y = e^{-x}$

$K = 3$ $y = 3e^{-x}$

$K = -1$ $y = -e^{-x}$



B.4 #3 Find parametric equations for the line through $(2,4,6)$ that is $\perp$ to the plane $x-y+3z=7$.

$\vec{n}=\langle 1,-1,3\rangle$ is perpendicular to the plane

Since the line is $\perp$ to the plane, $\vec{n}$ is parallel to the direction vector of the line.

The parametric equations are $x=2+t$

$y=4+(-1)\cdot t=4-t$

$z=6+3t$

#2 Find par. eqs. for the line through $(2,1,0)$ and perpendicular to both $\vec{i}+\vec{j}$ and $\vec{j}+\vec{k}$.

We are looking for $x=x_0+at$, $y=y_0+bt$, $z=z_0+ct$ where $\langle a,b,c\rangle$ is the direction vector of the line

$$(\vec{i}+\vec{j})\times(\vec{j}+\vec{k})=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&1&0\\0&1&1\end{vmatrix}=\begin{vmatrix}1&0\\1&1\end{vmatrix}\vec{i}-\begin{vmatrix}1&0\\0&1\end{vmatrix}\vec{j}+\begin{vmatrix}1&1\\0&1\end{vmatrix}\vec{k}$$

$=(1-0)\vec{i}-\vec{j}+\vec{k}=\vec{i}-\vec{j}+\vec{k}=\langle 1,-1,1\rangle$ ← direction vector of the line

$x=2+t$

$y=1-t$

$z=t$

Example where two lines intersect: (Found online) From RUTF University Learning Lab

L_1: $x = 5+t,\ y = 1-t,\ z = 8-3t$ $\rightarrow \langle 1, -1, -3 \rangle$

L_2: $x = 1+2s,\ y = -4+s,\ z = 8-2s$ $\rightarrow \langle 2, 1, -2 \rangle$

$\frac{2}{1} \neq \frac{1}{-1}$ $\Rightarrow$ not parallel

If the lines intersect)

$5+t = 1+2s$

$1-t = -4+s$

$5 = t+s$

$5 = \frac{2}{3}s + s$

$5 = \frac{5s}{3} \Rightarrow s = 3$

$8-3t = 8-2s$

$t = \frac{2}{3}s$

$t = 2$

* Verify that you get the same values for x, y, and z.

L_1: $x = 7,\ y = -1,\ z = 2$

Then check if we obtain the same point on L_2:

L_2: $x = 1+2(3) = 7$

$y = -4+3 = -1$

$z = 8-2(3) = 2$

The lines intersect at $(7, -1, 2)$

B.3 #2a Find the angle between the vectors $\vec{a} = \langle -2, 5 \rangle$, $\vec{b} = \langle 5, 12 \rangle$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a} \cdot \vec{b} = -2 \cdot 5 + 5 \cdot 12 = 50$$

$$|\vec{a}| = \sqrt{(-2)^2 + 25} = \sqrt{29}$$

$$|\vec{b}| = \sqrt{25 + 144} = 13$$

$$50 = \sqrt{29} \cdot 13 \cdot \cos\theta \Rightarrow \cos\theta = \frac{50}{\sqrt{29} \cdot 13}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{50}{\sqrt{29} \cdot 13}\right)}$$

B.6 #26 Draw a contour map of the function $f(x,y) = \sqrt[3]{x^2+y^2}$ showing several fixed curves.

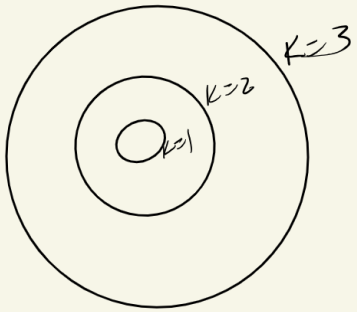
$$z = \sqrt[3]{x^2+y^2}$$

Note $z \geq 0$ so we should not pick negative values.

$z=1$: $1 = \sqrt[3]{x^2+y^2} \Rightarrow 1 = x^2+y^2$ circle centered at $(0,0)$ radius 1

$z=2$: $2 = \sqrt[3]{x^2+y^2} \Rightarrow 8 = x^2+y^2$ circle centered at $(0,0)$ radius $\sqrt{8}$

$z=3$: $3 = \sqrt[3]{x^2+y^2} \Rightarrow 27 = x^2+y^2$ circle centered at $(0,0)$ radius $\sqrt{27}$



#7 on PE: Find an eq of the plane through $(2,1,2)$, $(3,-8,6)$, and $(-2,-3,1)$.

$(2,1,2)$ = P, $(3,-8,6)$ = Q, $(-2,-3,1)$ = R

$\vec{PQ} = \langle 1, -9, 4 \rangle$

$\vec{PR} = \langle -4, -4, -1 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} = \begin{vmatrix} -9 & 4 \\ -4 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -9 \\ -4 & -4 \end{vmatrix}\vec{k}$$

$$= (9+16)\vec{i} - (-1+16)\vec{j} + (-4-36)\vec{k}$$

$$= 25\vec{i} - 15\vec{j} - 40\vec{k}$$

$ax + by + cz = d$

$25(x-2) - 15(y-1) - 40(z-2) = 0$

$25x - 15y - 40z - 35 + 80 = 0$

$$\boxed{25x - 15y - 40z = 45}$$

Exam 2 MTH-261

Prof. Leyda Almodóvar

Name: _____

Answer the questions in the spaces provided on thequestion sheets. If you run out of room for an answer, you may use an additional blank sheet. You must show your work on every problem. Answers with no work or justification shown will get no credit.

Question	Points	Score
1	10	
2	6	
3	8	
4	15	
5	12	
6	10	
7	10	
8	10	
9	8	
10	11	
Total:	100	

1. (10 points) Use the Chain Rule to find $\frac{\partial w}{\partial r}$ given $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$; when $r = 2$, $\theta = \frac{\pi}{2}$.

2. (6 points) Find the linearization $L(x, y)$ of the function $f(x, y) = x^2 e^y$ at the point $(1, 0)$.

3. (8 points) Given $f(x,y) = \frac{x}{y}$, $P = (2,1)$ and $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$:

   (a) Find the gradient of f.

   (b) Evaluate the gradient at the point P.

   (c) Find the rate of change of f at P in the direction of the vector $\vec{u}$.

4. (15 points) Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$.

5. (12 points) Use Lagrange multipliers to find the extreme values of the function $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

6. (10 points) (a) Sketch the region D bounded by $y = x, = x^3$, $x \geq 0$.

(b) Evaluate the double integral $\int\int_D (x^2 + 2y)\,\mathrm{d}A$.

7. (10 points) Evaluate the integral $\int\int_D x^2 y \, \mathrm{d}A$ by changing to polar coordinates, where D is the top half of the disk with center the origin and radius 5.

8. (10 points) Evaluate the triple integral $\int\int\int_E 6xy\,\mathrm{d}V$, where E lies under the plane $z = 1+x+y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

9. (8 points) Use cylindrical coordinates to evaluate $\int\int\int_E z\,\mathrm{d}V$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

10. (11 points) Use spherical coordinates to evaluate the integral $\int\int\int_E \sqrt{x^2+y^2+z^2}\,\mathrm{d}V$, where E lies above the cone $z=\sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

Review Exam 2

PE 10) $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$, where E lies above the cone $z=\sqrt{x^2+y^2}$ and between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$

$\rho^2=1$ $\qquad$ $\rho^2=4$ $\qquad$ $\rho^2=x^2+y^2+z^2$ $\qquad$ $\Rightarrow 1\le\rho\le 2$

$$\begin{cases} x=\rho\cos\theta\sin\phi \\ y=\rho\sin\theta\sin\phi \\ z=\rho\cos\phi \end{cases}$$

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV=\int_0^{\pi/4}\int_0^{2\pi}\int_1^2 \sqrt{\rho^2}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Need to find intersection between cone and spheres

$$\rho\cos\phi=\sqrt{\rho^2\cos^2\theta\sin^2\phi+\rho^2\sin^2\theta\sin^2\phi}=\sqrt{\rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)}$$

$$=\sqrt{\rho^2\sin^2\phi}=\rho\sin\phi$$

$$\Rightarrow \phi=\pi/4 \quad (0\le\phi\le\pi)$$

$$\int_0^{\pi/4}\sin\phi\,d\phi\cdot\int_0^{2\pi}d\theta\cdot\int_1^2\rho^3\,d\rho=\Big[-\cos\phi\Big]_0^{\pi/4}\cdot\Big[\theta\Big]_0^{2\pi}\cdot\Big[\frac{\rho^4}{4}\Big]_1^2$$

$$=\Big[-\cos\tfrac{\pi}{4}+\cos 0\Big]\cdot\Big[2\pi-0\Big]\cdot\Big[\tfrac{16}{4}-\tfrac{1}{4}\Big]$$

$$=\Big[-\tfrac{\sqrt{2}}{2}+1\Big]\cdot 2\pi\cdot\tfrac{15}{4}=\boxed{\tfrac{15\pi}{2}\left(1-\sqrt{2}/2\right)}$$

7) $\iint_D x^2 y \, dA$ D is the top half of the disk with center the origin and radius 5

$D = \{(x,y) \mid x^2 + y^2 \le 25, \; y \ge 0\}$

$D = \{(r,\theta) \mid 0 \le r \le 5, \; 0 \le \theta \le \pi\}$

$x = r\cos\theta$

$y = r\sin\theta$

$$\iint_D x^2 y \, DA = \int_0^{\pi} \int_0^5 (r\cos\theta)^2 r\sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \cdot \int_0^5 r^4 \, dr = \int_1^{-1} -u^2 \, du \cdot \left[\frac{r^5}{5}\right]_0^5$$

$u = \cos\theta$

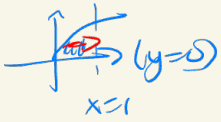
$du = -\sin\theta \, d\theta$

$$= \left[\frac{u^3}{3}\right]_{-1}^{1} \cdot \left[\frac{5^5}{5} - 0\right]$$

$$= \left[\frac{2}{3}\right] \cdot 5^4$$

$$= \frac{2}{3} \cdot 625 = \boxed{\frac{1250}{3}}$$

8) $\iiint_E 6xy\,dV$, where E lies under the plane $z = 1+x+y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.



$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{x},\ 0 \le z \le 1+x+y\}$$

$$0 \le y \le 1 \quad \text{or} \quad y^2 \le x \le 1$$

$$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xyz \Big]_0^{1+x+y} dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xy + 6x^2y + 6xy^2\,dy\,dx = \int_0^1 6x\frac{y^2}{2} + 6x^2\frac{y^2}{2} + 6x\frac{y^3}{3} \Big]_0^{\sqrt{x}} dx$$

$$= \int_0^1 3x(\sqrt{x})^2 + 3x^2(\sqrt{x})^2 + 2x(\sqrt{x})^3\,dx$$

$$= \int_0^1 3x^2 + 3x^3 + 2x^{5/2}\,dx = x^3 + \frac{3x^4}{4} + 2\cdot\frac{2}{7}x^{7/2} \Big]_0^1$$

$$= 1 + \frac{3}{4} + \frac{4}{7} = \boxed{\frac{65}{28}}$$

D.5 8) Find the volume of the solid that lies between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$.

$$V=\iiint_E 1\,dV$$

Switch to cylindrical coordinates: $z=r^2$

$r^2+z^2=2$

$z^2=2-r^2$

$z=\sqrt{2-r^2}$

$r^2\le z\le\sqrt{2-r^2}$

Intersection of paraboloid and sphere: $r^2+(r^2)^2=2$

$r^2+r^4-2=0$

$(r^2+2)(r^2-1)=0$

$r^2+2\neq 0$ (~~$r^2=-2$~~) $\quad r^2-1=0$, $r^2=1$, $r=1$

$$E=\{(r,\theta,z)\mid 0\le r\le 1,\ 0\le\theta\le 2\pi,\ r^2\le z\le\sqrt{2-r^2}\}$$

$$V=\iiint_E 1\,dV=\int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1 rz\Big]_{r^2}^{\sqrt{2-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 r(\sqrt{2-r^2}-r^2)\,dr\,d\theta=\int_0^{2\pi}d\theta\cdot\int_0^1 r\sqrt{2-r^2}-r^3\,dr$$

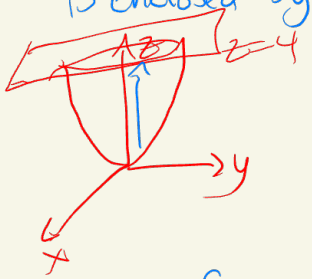
$u=2-r^2$, $du=-2r\,dr$, $-\frac{1}{2}du=r\,dr$

$$=\Big[\theta\Big]_0^{2\pi}\cdot\left[\left(-\frac{1}{2}\right)\int_2^1\sqrt{u}\,du-\left[\frac{r^4}{4}\right]_0^1\right]$$

$$=2\pi\cdot\left[-\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_2^1-\left[\frac{1}{4}-0\right]\right]$$

$$=2\pi\left[-\frac{1}{3}\cdot(1-2^{3/2})-\frac{1}{4}\right]$$

9) Use cylindrical coordinates to evaluate $\iiint_E z\,dV$, where E is enclosed by $z = x^2+y^2$ and the plane $z = 4$.



$x^2+y^2 = r^2$ $\quad r^2 \le z \le 4$

$z = r^2$

$\Rightarrow r^2 = 4 \Rightarrow 0 \le r \le 2$

$$E = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le 4\}$$

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{z^2}{2}\cdot r\Big]_{r^2}^4 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 8r - \frac{r^5}{2}\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 8r - \frac{r^5}{2}\,dr$$

$$= \theta\Big]_0^{2\pi}\cdot\left[4r^2 - \frac{r^6}{12}\right]_0^2 = 2\pi\left[4\cdot 4 - \frac{64}{12}\right] = \boxed{\frac{64\pi}{3}}$$

4) Find local max/min and saddle point(s) of

$f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$

$f_x = 3x^2 - 6x - 9 = 0$ and $f_y = 3y^2 - 6y = 0$

$3(x^2 - 2x - 3) = 0$ $\qquad$ $3y(y-2) = 0$

$(x-3)(x+1) = 0$ $\qquad$ $y = 0, y = 2$

$x = 3, x = -1$

C.P.s: $(3,0), (3,2), (-1,0), (-1,2)$

$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$f_{xx} = 6x - 6$, $f_{yy} = 6y - 6$, $f_{xy} = 0$

$D(3,0) = f_{xx}(3,0) f_{yy}(3,0) - 0^2 = 12 \cdot (-6) = -72 < 0$

$\Rightarrow (3,0)$ is saddle point

$D(3,2) = f_{xx}(3,2) f_{yy}(3,2) - 0^2$

$= 12 \cdot (6) = 72 > 0 \Rightarrow$ local min. at $(3,2)$

$D(-1,0) = f_{xx}(-1,0) \cdot f_{yy}(-1,0) - 0^2 = (-12) \cdot (-6) = 72 > 0 \Rightarrow$ local max. at $(-1,0)$

$D(-1,2) = f_{xx}(-1,2) f_{yy}(-1,2) - 0^2$

$= (-12) \cdot (6) = -72 < 0 \Rightarrow$ Saddle point at $(-1,2)$

5) Use Lagrange multipliers to find extreme values

$f(x,y) = xy$ subject to constraint $4x^2 + y^2 = 8$

"g"

$\nabla f = \lambda \nabla g$

$\nabla g = \vec{0}$

$\langle y, x \rangle = \lambda \langle 8x, 2y \rangle$

$\langle 8x, 2y \rangle = \langle 0, 0 \rangle$

$(0,0)$

$f(0,0) = 0$

$(0,0,0)$

$y = \lambda \cdot 8x$ and $x = \lambda \cdot 2y$

$\frac{y}{8x} = \lambda$ and $\frac{x}{2y} = \lambda$

$\frac{y}{8x} = \frac{x}{2y} \Rightarrow 2y^2 = 8x^2$

$y^2 = 4x^2$ $\rightarrow$ $4x^2 + 4x^2 = 8$

$8x^2 = 8$

$x = \pm 1$

If $x = -1$, $y^2 = 4$ $\Rightarrow y = -2$ or $y = 2$

If $x = 1$, $y^2 = 4$ $\Rightarrow y = -2$ or $y = 2$

$(-1,-2)$, $(-1,2)$

$(1,-2)$, $(1,2)$

$f(-1,-2) = f(1,2) = 2$

$f(-1,2) = f(1,-2) = -2$

Abs. min $(-1,2,-2)$ and $(1,-2,-2)$

Abs. max $(-1,-2,2)$ and $(1,2,2)$

Exam 3 MTH-261

Prof. Leyda Almodóvar

Name: _____

Answer the questions in the spaces provided on thequestion sheets. If you run out of room for an answer, you may use an additional blank sheet. You must show your work on every problem. Answers with no work or justification shown will get no credit.

Question	Points	Score
1	8	
2	8	
3	9	
4	9	
5	9	
6	10	
7	10	
8	12	
9	10	
10	7	
11	8	
Total:	100	

1. (8 points) Evaluate the line integral $\int_C 3x^2 \, \mathrm{d}s$, where C is the line segment from $(1, 2)$ to $(3, -1)$.

2. (8 points) Given $\vec{F}(x,y) = xy^2\vec{i} - x^2\vec{j}$, evaluate the line integral $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$, where C is given by the vector function $\vec{r}(t) = t^3\vec{i} + t^2\vec{j}$, $0 \leq t \leq 1$.

3. (9 points) Given $\vec{F}(x,y) = (3+2xy^2)\vec{i} + 2x^2y\vec{j}$, find a function f such that $\vec{F} = \nabla f$ and use it to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve C: the arc of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

4. (9 points) Find the curl and the divergence of the vector field $\vec{F}(x, y, z) = xy^2z^2\vec{i} + x^2yz^2\vec{j} + x^2y^2z\vec{k}$.

5. (9 points) Use Green's Theorem to evaluate the line integral $\int_C (y + e^{\sqrt{x}})\,\mathrm{d}x + (2x + \cos y^2)\,\mathrm{d}y$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

6. (10 points) Find an equation of the tangent plane to the parametric surface $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$ at the point $(5, 2, 3)$.

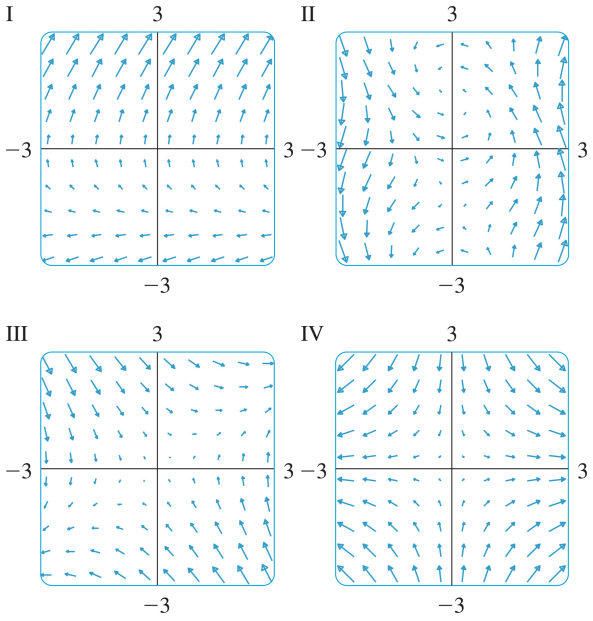
7. (10 points) Find the area of the surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}})$, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

8. (12 points) Evaluate the surface integral $\int\int_S xyz\,\mathrm{d}S$, S is the part of the plane $z = 1+2x+3y$ that lies above the rectangle $0 \le x \le 3$, $0 \le y \le 2$.

9. (10 points) Let $\vec{F}(x, y, z) = \langle ze^y, x\cos y, xz\sin y\rangle$ and let S be the hemisphere $x^2 + y^2 + z^2 = 16$, $y \leq 0$, oriented in the direction of the positive y-axis. Use Stokes' Theorem to evaluate $\int\int_S$ curl $\vec{F} \cdot \mathrm{d}\vec{S}$.

10. (7 points) Let $\vec{F}(x,y,z) = (2x^3 + y^3)\vec{i} + (y^3 + z^3)\vec{j} + 3y^2z\vec{k}$ and S be the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane. Use the Divergence Theorem to calculate the flux of $\vec{F}$ across S.

11. (8 points) Match the vector fields F with the plots labeled I-IV. Give **two** reasons for each choice.



(a) $\vec{F}(x, y) = \langle x, -y \rangle$

(b) $\vec{F}(x, y) = \langle y, x - y \rangle$

(c) $\vec{F}(x, y) = \langle y, y + 2 \rangle$

(d) $\vec{F}(x, y) = \langle \cos(x + y), x \rangle$

Review Exam 3

P.E. 1) $\int_C 3x^2 ds$, where C is the line segment from $(\underset{x_0}{1}, \underset{y_0}{2})$ to $(\underset{x_1}{3}, \underset{y_1}{-1})$.

$$x = (1-t)x_0 + tx_1 = (1-t)\cdot 1 + 3t = 2t+1$$
$$y = (1-t)y_0 + ty_1 = (1-t)\cdot 2 - t = -3t+2, \quad 0 \le t \le 1$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(2)^2 + (-3)^2}\, dt = \sqrt{13}\, dt$$

$$\int_C 3x^2 ds = \int_0^1 3(2t+1)^2 \cdot \sqrt{13}\, dt = 3\sqrt{13}\int_0^1 4t^2 + 4t + 1\, dt$$

$$= 3\sqrt{13}\cdot\left[\frac{4t^3}{3} + 2t^2 + t\right]_0^1 = 3\sqrt{13}\left[\frac{4}{3} + 2 + 1\right] = 3\sqrt{13}\left[\frac{13}{3}\right]$$

$$= \boxed{13\sqrt{13}}$$

2) $\vec{F}(x,y) = xy^2\vec{i} - x^2\vec{j}$, find $\int_C \vec{F}\cdot d\vec{r}$, where C is given by $\vec{r}(t) = t^3\vec{i} + t^2\vec{j}$, $0 \le t \le 1$.

$$\vec{F}(\vec{r}(t)) = t^3 \cdot t^4\vec{i} - t^6\vec{j} = t^7\vec{i} - t^6\vec{j}$$

$$\vec{r}\,'(t) = 3t^2\vec{i} + 2t\vec{j}$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 3t^9 - 2t^7$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 3t^9 - 2t^7\, dt = \frac{3t^{10}}{10} - \frac{2t^8}{8}\Big]_0^1$$

$$= \frac{3}{10} - \frac{1}{4} = \boxed{\frac{1}{20}}$$

9) $\vec{F}(x,y,z) = \langle ze^y, x\cos y, x z \sin y \rangle$

S is the hemisphere $x^2+y^2+z^2=16$, $y \le 0$, oriented in the direction of the positive y-axis.

Use Stokes' Theorem to find $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$.

The boundary C is the circle $x^2+z^2=16$, $y=0$

C needs to be oriented counterclockwise (looking at it from the positive y-axis)

$x = r\cos(t) = 4\cos t$ $\quad 0 \le t \le 2\pi$ $\quad \cos(-t)=\cos t$

$z = r\sin(t) = -4\sin t$ $\quad \sin(-t)=-\sin t$

$\vec{r}(t) = 4\cos t\,\vec{i} - 4\sin t\,\vec{k}$ $\quad (y=0)$

$\vec{r}\,'(t) = -4\sin t\,\vec{i} - 4\cos t\,\vec{k}$

$\vec{F}(\vec{r}(t)) = -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 0\vec{k}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 16\sin^2 t + 0 + 0 = 16\sin^2 t$

$$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \int_0^{2\pi} 16\sin^2 t\,dt = 8\int_0^{2\pi} 1-\cos 2t\,dt$$

$$= 8\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = 8\left[2\pi - \frac{\sin 4\pi}{2} - 0 + \frac{\sin 0}{2}\right] = \boxed{16\pi}$$

10) $\vec{F}(x,y,z) = (2x^3+y^3)\vec{i} + (y^3+z^3)\vec{j} + 3y^2z\vec{k}$

S is the surface of the solid bounded by $z = 1-x^2-y^2$ and the xy-plane. Use the Divergence Theorem to find the flux of $\vec{F}$ across S.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x^2 + 3y^2 + 3y^2 = 6x^2 + 6y^2$$

$$= \iiint_E 6x^2 + 6y^2\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 6r^2 \cdot r\, dz\, dr\, d\theta$$

($r^2 = x^2+y^2$)

$$= \int_0^{2\pi}\int_0^1 6r^3 z\Big]_0^{1-r^2} dr\, d\theta = \int_0^{2\pi}\int_0^1 6r^3\left[1-r^2\right] dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 6r^3 - 6r^5\, dr\, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^1 6r^3 - 6r^5\, dr$$

$$= \left[\theta\right]_0^{2\pi}\cdot\left[\frac{6r^4}{4} - r^6\right]_0^1 = 2\pi\left[\frac{3}{2} - 1\right] = \boxed{\pi}$$

7) Find the area of the surface $z = \frac{2}{3}(x^{3/2}+y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$.
$= g(x,y)$

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \frac{2}{3}(x^{3/2}+y^{3/2})\vec{k}$

$\vec{r}_x = \vec{i} + x^{1/2}\vec{k}$

$\vec{r}_y = \vec{j} + y^{1/2}\vec{k}$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & x^{1/2} \\ 0 & 1 & y^{1/2} \end{vmatrix} = \begin{vmatrix} 0 & x^{1/2} \\ 1 & y^{1/2} \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & x^{1/2} \\ 0 & y^{1/2} \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{k}$$

$$= -x^{1/2}\vec{i} - y^{1/2}\vec{j} + \vec{k} = \langle -g_x, -g_y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(-x^{1/2})^2 + (-y^{1/2})^2 + 1^2} = \sqrt{x+y+1}$$

$$A = \iint_D |\vec{r}_x \times \vec{r}_y| \, dA = \int_0^1 \int_0^1 \sqrt{x+y+1} \, dy \, dx = \int_0^1 \int_{x+1}^{x+2} \sqrt{u} \, du \, dx$$

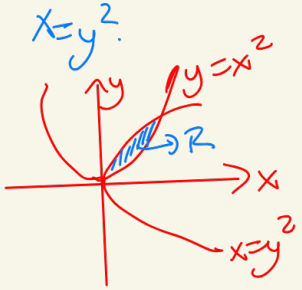
$u = x+y+1$, $du = dy$

$$= \int_0^1 \frac{2}{3} u^{3/2} \Big]_{x+1}^{x+2} dx = \frac{2}{3}\int_0^1 (x+2)^{3/2} - (x+1)^{3/2} \, dx$$

$$= \frac{2}{3}\left[\frac{2}{5}(x+2)^{5/2} - \frac{2}{5}(x+1)^{5/2}\right]_0^1 = \frac{4}{15}\left[3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right]$$

$$= \boxed{\frac{4}{15}\left[3^{5/2} - 2^{7/2} + 1\right]}$$

5) Use Green's Theorem to evaluate $\int_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$ where C is the boundary of the region enclosed by $y=x^2$ and $x=y^2$.



$R=\{(x,y) \mid x^2 \le y \le \sqrt{x},\ 0 \le x \le 1\}$

$x^2 = \sqrt{x}$

$x^2 - \sqrt{x} = 0$

$x^{1/2}(x^{3/2}-1)=0$

$x=0,\ x=1$

$$\int_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(2x+\cos y^2) - \frac{\partial}{\partial y}(y+e^{\sqrt{x}})\right] dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2 - 1 \, dy\,dx$$

$$= \int_0^1 y \Big]_{x^2}^{\sqrt{x}} dx = \int_0^1 x^{1/2} - x^2 \, dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big]_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

6) Find an equation of the tangent plane to $x=u^2+1$, $y=v^3+1$, $z=u+v$ at the point $(5,2,3)$.

$$\vec{r}(u,v) = (u^2+1)\vec{i} + (v^3+1)\vec{j} + (u+v)\vec{k}$$

$$\vec{r}_u = 2u\vec{i} + 1\vec{k}$$

$$\vec{r}_v = 3v^2\vec{j} + 1\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 3v^2 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 2u & 1 \\ 0 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 2u & 0 \\ 0 & 3v^2 \end{vmatrix}\vec{k}$$

$$= -3v^2\vec{i} - 2u\vec{j} + 6uv^2\vec{k}$$

Find values of u and v corresponding to $(5,2,3)$

$5 = u^2+1$, $2 = v^3+1$, $3 = u+v$

$1 = v^3$ → $1 = v$ → $3 = u+1$ → $u = 2$

normal vector is $\langle -3, -4, 12 \rangle$

Tangent plane

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$-3(x-5) - 4(y-2) + 12(z-3) = 0$$

$$\boxed{-3x - 4y + 12z = 13}$$

3) $\vec{F}(x,y) = (3+2xy^2)\vec{i} + 2x^2y\vec{j}$, find f such that $\vec{F} = \nabla f$ and use it to evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the arc of $y = \frac{1}{x}$ from $(1,1)$ to $(4,\frac{1}{4})$.

$f_x = 3+2xy^2 \Rightarrow f(x,y) = 3x + x^2y^2 + g(y)$

$f_y = 2x^2y \Rightarrow f_y = 0 + 2x^2y + g'(y) \Rightarrow g'(y) = 0$

$g(y) = K$, constant

$f(x,y) = 3x + x^2y^2 + K$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(4, 1/4) - f(1,1)$

$= 3\cdot 4 + 4^2\cdot(\frac{1}{4})^2 - [3+1]$

$= 12 + 1 - 4 = \boxed{9}$

8) $\iint_S xyz\,dS$, S is the part of the plane $z = 1+2x+3y$ $= g(x,y)$ that lies above the rectangle $0 \le x \le 3$, $0 \le y \le 2$

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (1+2x+3y)\vec{k}$

$\vec{r}_x = \vec{i} + 2\vec{k}$ $\qquad \vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$

$\vec{r}_y = \vec{j} + 3\vec{k}$ $\qquad = \langle -2, -3, 1 \rangle$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = \sqrt{14}$$

$$\iint_S xyz\,dS = \int_0^3 \int_0^2 xy(1+2x+3y)\sqrt{14}\,dy\,dx$$

$$= \sqrt{14} \int_0^3 \int_0^2 xy + 2x^2y + 3xy^2\,dy\,dx$$

$$= \sqrt{14} \int_0^3 \left[\frac{xy^2}{2} + x^2y^2 + xy^3 \right]_0^2 dx$$

$$= \sqrt{14} \int_0^3 2x + 4x^2 + 8x\,dx = \sqrt{14} \left[5x^2 + \frac{4x^3}{3} \right]_0^3$$

$$= \sqrt{14} \cdot [45 + 36] = \boxed{81\sqrt{14}}$$